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ANALYSIS OF SLENDER REINFORCED CONCRETE COLUMNS BASED ON A NONLINEAR MIXED FORMULATION WITH ANALYTICAL INTERPOLATION FUNCTIONS

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Abstract. The design of concrete columns requires two main steps: global and local analysis. Generally, the global analysis is performed using a simplified model considering the γ_z coefficient with a stiffness reduction of the structural elements to simulate the nonlinear behavior. In contrast, the local analysis demands a more rigorous process, particularly for slender columns. Typically, column analysis usually is conducted using the Finite Element Method, with appropriate discretization to accurately capture the nonlinear behavior and effects P-D and P-d. In this context, the local modelling of columns using a mixed element formulation with analytical interpolation functions can avoid the need for intense discretization while effectively capturing the curvature along the element. evaluates a design methodology for slender and very slender reinforced concrete columns based on a mixed element formulation, which employs analytical interpolation functions to approximate displacements and internal forces and uses a corotational total Lagrangian description. The P-δ effects are naturally considered in this approach. The proposed numerical application clearly demonstrates the capacity of the formulation to simulate the nonlinear local analysis of reinforced concrete columns using just one element and to accurately capture the curvature and bending moments along the column in the design process.

Keywords: Slender reinforced concrete columns, mixed element formulation, analytical interpolation functions, nonlinear analysis, single element.

1 Introduction

In general, the design of reinforced concrete columns involves two steps [1]. First, a global analysis of the building is performed, considering nonlinearities through simplified methods, such as the γ_z and stiffness reduction of structural elements. The second step is a local analysis of the column bay, in which geometric and material nonlinearities should be considered using more refined methods. For the NBR 6118:2023 [1], the local model of columns with a slenderness $\lambda \leq 90$, the nonlinear analysis can be considered by the simplified "standard column method" associated to "approximated curvature" or "approximated kappa rigidity" to simulate the material behavior. However, in cases that the slenderness is high ($\lambda > 140$), NBR 6118:2023 [1] demands the use of the "General Method", i.e., the analysis of the column must consider the material and geometric nonlinearities through a complete nonlinear analysis with a refined discretization using a numerical solution as the finite element method (FEM).

The discretization is fundamental in a nonlinear analysis using FEM. This limitation of the usual displacement-based elements can be overcome using higher-order displacement interpolation functions [2], stability functions [3], flexibility-based [4] or mixed elements [5]. Recently, Rodrigues et al. [6] proposed a 3D mixed formulation, implemented in the OpenSees [7] to simulate frames and columns with a single element. The formulation uses analytical interpolation functions based on the solution of the equilibrium of a deformed infinitesimal element [8,9].

Therefore, this work evaluates the formulation proposed by Rodrigues et al. [6] to analyze slender columns

(λ >140) using just one element. The material nonlinearity is simulated through the N-M-1/r diagram for the column bay and is associated with the mixed formulation. Thus, it is possible to simulate the local analysis (P-δ) effect by considering both geometric and material nonlinear behavior without discretization, while accurately capturing the curvature and bending moments along the column during the design process.

2 Element Formulation

2.1 Corotational total Lagrangian framework

The proposed formulation employs a corotational approach [5], Fig. 1. The deformed configuration $\widehat{\boldsymbol{D}}$, is composed by the u, v nodal translations, and the cross-section rotation θ, all related to the global x–y axes. However, the motions that effectively produce internal forces are generated by natural displacements ($\boldsymbol{D_n}$), an elongation, u_n, with respect to the initial configuration, and two relative rotations to the corotated configuration, θ_{nz1} and θ_{nz2}, according eq. (1). Similarly, the vectors of nodal forces and moments in the global system, $\widehat{\boldsymbol{P}}$, and the forces ($\boldsymbol{P}_n$) generated by natural displacements, are given in eq. (2).

$$\widehat{\boldsymbol{D}} = [u_1 \quad v_1 \quad \theta_{z1} \quad u_2 \quad v_2 \quad \theta_{z2}]^T \qquad \boldsymbol{D_n} = [u_n \quad \theta_{nz1} \quad \theta_{nz2}]^T, \tag{1}$$

$$\widehat{\boldsymbol{P}} = \{f_{x1} \quad f_{y1} \quad M_{z1} \quad f_{x2} \quad f_{y2} \quad M_{z2}\}^T \qquad \boldsymbol{P}_n = \{P \quad M_{z1} \quad M_{z2}\}^T. \tag{2}$$

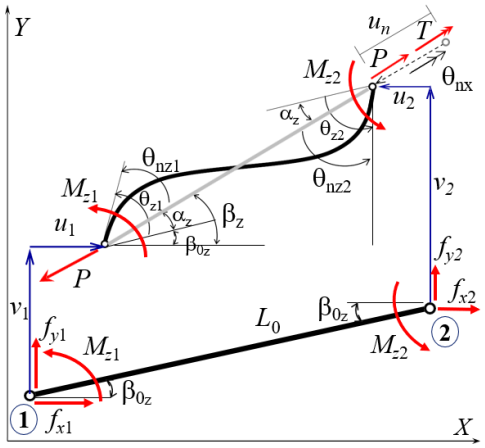


Figure 1. Displacement components in global and natural systems (adapted from Rodrigues et al. [6])

The displacement, force, and stiffness transformations between natural and global systems are:

$$\delta \boldsymbol{D_n} = \boldsymbol{T}\delta\widehat{\boldsymbol{D}} \qquad \widehat{\boldsymbol{P}} = \boldsymbol{T}^T\boldsymbol{P}_n \qquad \widehat{\boldsymbol{K}} = \boldsymbol{T}^T\boldsymbol{K}_n\boldsymbol{T} + \boldsymbol{K}_G, \tag{3}$$

where $\boldsymbol{T}$ is a transformation matrix connecting the global and natural systems, $\boldsymbol{K}_G$ is the external geometric stiffness matrix [4], $\boldsymbol{K}_n$ is the tangent stiffness matrix in the natural system, and $\widehat{\boldsymbol{K}}$ is the tangent stiffness matrix in the global system. Usually, the $\boldsymbol{K}_n$ natural system stiffness matrix employs Hermitian interpolation functions.

2.2 Beam kinematics

The displacement field of a generic point *P(x, y)* considering the Euler–Bernoulli theory, is expressed as:

$$u_P = u - yv' \qquad v_P = v, \tag{4}$$

where u_P and v_P are the axial and transverse displacement of the point P, while variables u and v of the cross-section center of gravity. The superscript tilde identifies the displacement derivative with respect to variable x along the element length. From the displacement field, the Green–Lagrange axial strain tensor (ε) is:

$$\varepsilon = u_P{}' + \frac{1}{2}(v_P{}')^2 = u' - yv'' + \frac{1}{2}[(v')^2] \rightarrow \hat{\boldsymbol{\epsilon}} = [1 \quad -y]\begin{bmatrix} u' + \frac{1}{2}[(v')^2] \\ v'' \end{bmatrix} = \boldsymbol{Y}\hat{\boldsymbol{d}}. \tag{5}$$

2.3 Cross-section constitutive relation

The element was implemented in the OpenSees [7], that uses the fiber method for nonlinear analysis of beam-columns. Thus, the constitutive relation for an element is determined by integrating the stress–strain relations over the cross-section. The stress resultant vector $\boldsymbol{S_\Sigma}$ is given by

$$\boldsymbol{S}_\Sigma = \int_{A_0} \boldsymbol{Y}^T \boldsymbol{\sigma}(\boldsymbol{\epsilon}) dA = [1 \quad -y]^T \sigma, \tag{6}$$

where σ is the corresponding stress. Using the strain–displacement relation $\boldsymbol{\epsilon} = \boldsymbol{Yd}$, the section tangent stiffness matrix is expressed as:

$$\boldsymbol{K}_s = \frac{\partial \boldsymbol{S}_\Sigma}{\partial \boldsymbol{d}} = \int_{A_0} \boldsymbol{Y}^T \boldsymbol{E} \boldsymbol{Y} dA = \int_{A_0} \begin{bmatrix} E & -yE \\ -yE & y^2E \end{bmatrix} dA \tag{7}$$

with E corresponding to the Young's modulus.

2.4 Analytical interpolation functions

The study of a deformed infinitesimal beam-column element subjected to a distributed load $q(x)$, Fig. 2, leads to the equilibrium expressions eq. (8) to eq. (10).

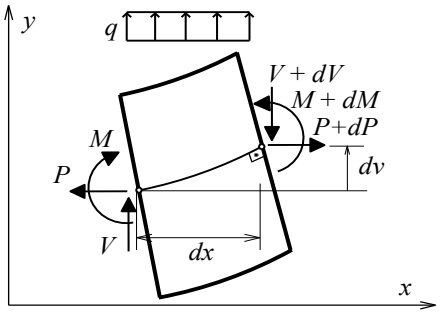


Figure 2. Equilibrium of a deformed beam-column element (adapted from [8])

$$\sum F_x \rightarrow \frac{dP}{dx} = 0, \tag{8}$$

$$\sum F_y \rightarrow -dV + q(x)dx = 0 \rightarrow \frac{dV(x)}{dx} = q(x), \tag{9}$$

$$\sum M_o \rightarrow dM - (V + dV)dx - (P + dP).dv + q(x)\frac{dx^2}{2} = 0. \tag{10}$$

where $v(x)$ is the infinitesimal element transverse displacement, $V(x)$is the vertical component of the force acting on the cross-section, P is the horizontal component, and $M(x)$ is the bending moment. Disregarding the quadratic differential terms, and the approximate relation between bending moment and curvature, $M(x) = EI\, d\theta/dx$, the equilibrium relation of a deformed infinitesimal beam-column element can be represented by,

$$\frac{d^2M(x)}{dx^2} - P\frac{d^2v(x)}{dx^2} = \frac{d^4v(x)}{dx^4} - \frac{P}{EI_z}\frac{d^2v(x)}{dx^2} = q(x). \tag{11}$$

The homogeneous solution, $q(x) = 0$, of the differential equation corresponds to the final solution in the direct stiffness method. For $\mu = \sqrt{P/EI_z}$, the solution of the differential equation is given by,

$$v(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} + c_3 x + c_4, \tag{12}$$

$$\theta(x) = c_1 \mu e^{\mu x} - c_2 \mu e^{-\mu x} + c_3. \tag{13}$$

As the deformed configuration of an isolated element can be described by interpolating the nodal displacements, Fig. 3:

$$u_o(x) = N_1^u(x)d_1' + N_4^u(x)d_4', \tag{14}$$

$$v_0(x) = N_2^v(x)d_2' + N_3^v(x)d_3' + N_5^v(x)d_5' + N_6^v(x)d_6', \tag{15}$$

$$\theta z(x) = N_2^{\theta z}(x)d_2' + N_3^{\theta z}(x)d_3' + N_5^{\theta z}(x)d_5' + N_6^{\theta z}(x)d_6'. \tag{16}$$

where N_1^u, N_4^u are the axial, and $N_2^v, N_3^v, N_5^v, N_6^v$ the transverse displacement interpolation functions.

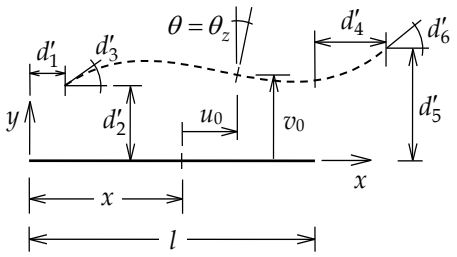


Figure 3. Deformed configuration of an isolated element (adapted from [8])

As the formulation follows the corotational approach for columns design, $P < 0$ with $\mu = \sqrt{-P/EI_z}$, the transversal displacements $d_1' = d_2' = d_5' = 0$. Therefore, the necessary interpolation functions are:

$$N_4^u(x) = x/L, \tag{17}$$

$$N_3^v = -\frac{sin(\mu x) - sin(L\mu) + sin\big(\mu(L-x)\big) + \mu x + L\mu cos(L\mu) - \mu x cos(L\mu) - L\mu cos\big(\mu(L-x)\big)}{\mu(2\,cos(L\mu) + L\mu sin(L\mu) - 2)}, \tag{18}$$

$$N_6^v = \frac{sin(\mu x) - sin(L\mu) + sin\big(\mu(L-x)\big) + L\mu - \mu x - L\mu cos(\mu x) + \mu x cos(L\mu)}{\mu(2\,cos(L\mu) + L\mu sin(L\mu) - 2)}, \tag{19}$$

$$N_3^{\theta z} = \frac{cos(L\mu) - cos(\mu x) + cos\big(\mu(L-x)\big) + L\mu sin\big(\mu(L-x)\big) - 1}{2\,cos(L\mu) + L\mu sin(L\mu) - 2}, \tag{20}$$

$$N_6^{\theta z} = \frac{cos(L\mu) + cos(\mu x) - cos\big(\mu(L-x)\big) + L\mu sin(\mu x) - 1}{2\,cos(L\mu) + L\mu sin(L\mu) - 2}, \tag{21}$$

MATLAB and C open-source codes with these analytical functions are available in Rodrigues et al. [10,11].

Finally, the displacement field as the function of natural system nodal displacements ($\boldsymbol{D_n}$) is given by:

$$u = \boldsymbol{N}_u^T \boldsymbol{D}_n = [N_4^u \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]\boldsymbol{D}_n, \tag{22}$$

$$v = \boldsymbol{N}_v^T \boldsymbol{D}_n = [0 \quad N_3^v \quad N_6^v \quad 0 \quad 0 \quad 0]\boldsymbol{D}_n, \tag{23}$$

$$\theta_z = \boldsymbol{N}_{\theta_z}^T \boldsymbol{D}_n = [0 \quad N_3^{\theta z} \quad N_6^{\theta z} \quad 0 \quad 0 \quad 0]\boldsymbol{D}_n. \tag{24}$$

The strain–displacement relations are then written using the analytical shape functions with an infinitesimal change in $\hat{\boldsymbol{\epsilon}}$ as follows:

$$\delta\hat{\boldsymbol{\epsilon}} = \boldsymbol{Y}\delta\hat{\boldsymbol{d}} = \boldsymbol{Y}\boldsymbol{N}_{\delta\hat{d}_1}\boldsymbol{N}_{\delta\hat{d}_2}\delta\boldsymbol{D}_n, \tag{25}$$

$$\boldsymbol{N}_{\delta\hat{d}1} = \begin{bmatrix} 1 & v' & 0 \\ 0 & 0 & v'' \end{bmatrix} \qquad \boldsymbol{N}_{\delta\hat{d}_2} = \begin{bmatrix} N_4^u & 0 & 0 \\ 0 & N_3^{v'} & N_6^{v'} \\ 0 & N_3^{v''} & N_6^{v''} \end{bmatrix}, \tag{26}$$

Furthermore, to solve the system presented in eq. (8) to (10), the natural boundary conditions are $N(L) = P,\ M(0) = -M_{z1},\ M(L) = M_{z2}$, thus the stress resultant fields are obtained as

$$N(x) = P \qquad M_z(x) = v(x)P + \left(\frac{x}{L} - 1\right)M_{z1} + \frac{x}{L}M_{z2}. \tag{27}$$

The P-δ effects are considered in the moment fields based on the interpolated transverse displacement v.

The generalized stress resultant internal force field within element S can be expressed as follows:

$$\boldsymbol{S} = \begin{bmatrix} N \\ M_z \end{bmatrix} = \begin{bmatrix} N_A & 0 & 0 \\ v & N_{z1} & N_{z2} \end{bmatrix} \begin{bmatrix} P \\ M_{z1} \\ M_{z2} \end{bmatrix} = \boldsymbol{N}_{D1}\boldsymbol{P}_s, \tag{28}$$

$$N_{z1} = N_{y1} = \frac{x}{L} - 1 \quad N_{z2} = N_{y2} = \frac{x}{L} \quad N_A = N_W = N_T = 1. \tag{29}$$

2.5 Mixed formulation

The mixed formulation is based on the Hellinger–Reissner variational principle [5,12] composed by the element equilibrium equation ($\boldsymbol{g}$), from the virtual work principle, the element strain–displacement compatibility ($\boldsymbol{V}$), and section equilibrium ($\boldsymbol{U}$) related to the generalized stress resultant vector,

$$\begin{cases} \boldsymbol{g} = \int_{l_0} \boldsymbol{N}_{\delta\hat{d}2}^T \boldsymbol{N}_{\delta\hat{d}1}^T \boldsymbol{S}\, dx + \int_{l_0} \boldsymbol{N}_{D2}^T (\hat{\boldsymbol{d}} - \boldsymbol{d})\, dx - \boldsymbol{P}_{ext} = \boldsymbol{0} \\ \boldsymbol{V} = \int_{l_0} \boldsymbol{N}_{D1}^T (\hat{\boldsymbol{d}} - \boldsymbol{d})\, dx = \boldsymbol{0} \\ \boldsymbol{U} = \boldsymbol{S}_\Sigma - \boldsymbol{S} = \boldsymbol{0}, \end{cases} \tag{30}$$

The tangent stiffness matrix $\boldsymbol{K}_n$ in the natural system is obtained through the linearization of the equations $\boldsymbol{g}$, $\boldsymbol{V}$ and $\boldsymbol{U}$ [5,12] as,

$$\boldsymbol{K}_n = \left(\boldsymbol{K}_g + \boldsymbol{G}_2^T + \boldsymbol{G}_2 - \boldsymbol{H}_{22}\right) + (\boldsymbol{G}_1 + \boldsymbol{M}_d - \boldsymbol{H}_{12})^T \boldsymbol{H}_{11}^{-1} (\boldsymbol{G}_1 + \boldsymbol{M}_d - \boldsymbol{H}_{12}). \tag{31}$$

$$\boldsymbol{G}_1 = \int_{l_0} \boldsymbol{N}_{D1}^T \boldsymbol{N}_{\delta\hat{d}1} \boldsymbol{N}_{\delta\hat{d}2}\, dx; \quad \boldsymbol{G}_2 = \int_{l_0} \boldsymbol{N}_{D2}^T \boldsymbol{N}_{\delta\hat{d}1}\, \boldsymbol{N}_{\delta\hat{d}2} dx\,. \tag{32}$$

$$\boldsymbol{H}_{11} = \int_{l_0} \boldsymbol{N}_{D1}^T \boldsymbol{f} \boldsymbol{N}_{D1}\, dx;\ \boldsymbol{H}_{12} = \int_{l_0} \boldsymbol{N}_{D1}^T \boldsymbol{f} \boldsymbol{N}_{D2}\, dx;\ \boldsymbol{H}_{22} = \int_{l_0} \boldsymbol{N}_{D2}^T \boldsymbol{f} \boldsymbol{N}_{D2}\, dx;\ \boldsymbol{f} = \boldsymbol{K}_s^{-1} \tag{33}$$

$$\boldsymbol{K}_g = \int_{l_0} \boldsymbol{N}_{\delta\hat{d}2}^T \boldsymbol{G}\, \boldsymbol{N}_{\delta\hat{d}2} dx; \quad \boldsymbol{G} = \begin{bmatrix} P & 0 & 0 \\ 0 & 0 & M_z \\ 0 & M_z & 0 \end{bmatrix} \tag{34}$$

$$\boldsymbol{M}_d = \begin{bmatrix} 0 & N_3^v(\hat{\boldsymbol{d}} - \boldsymbol{d})[2] & N_6^v(\hat{\boldsymbol{d}} - \boldsymbol{d})[2] \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boldsymbol{N}_{D2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & PN_3^v & PN_6^v \end{bmatrix} \tag{35}$$

The complete element state determination procedure is developed in [5,12].

2.6 Material nonlinear formulation

For the design of reinforced concrete columns using the general method the moment-curvature relations (N-M-1/r diagram) for each element should be obtained [13]. The curvature of a reinforced concrete cross-section can be calculated considering the concrete and steel strains, ε_c and ε_s, and the effective height d, Fig. 4. Thus, the curvature can be related to the acting bending moment through the stiffness EI,

$$\frac{1}{r} = \frac{\varepsilon_c + \varepsilon_s}{d} \rightarrow M = EI\frac{1}{r}. \tag{36}$$

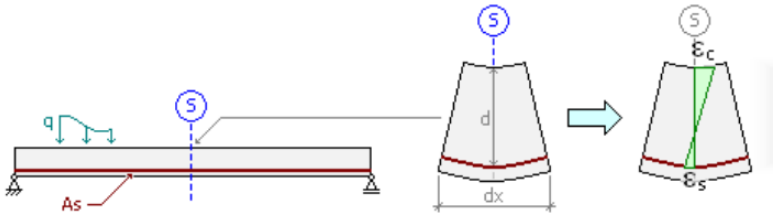


Figure 4. Curvature of a reinforced concrete cross-section (adapted from [13])

Instead of stiffness EI obtained for each element (Fig. 5a), a more recommendable and safer procedure is to consider the linearization of the diagram up to the ultimate strength of the cross-section for the entire column bay, considering a defined steel reinforcement and an axial load level, as illustrated in Fig. 5b, returning a equivalent secant stiffness EI_{sec}, that should be used in the stiffness matrix to represent the reinforced concrete column.

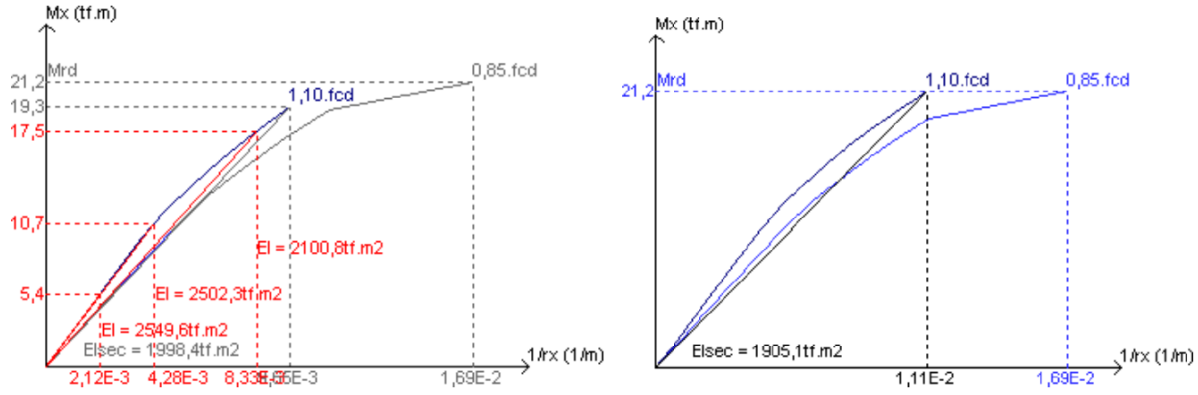


Figure 5. N-M-1/r diagram for a reinforced concrete cross-section (adapted from [13])

3 Numerical Application

The example studies a column with a rectangular cross-section 25x60 cm, length L = 14.40 m, and slenderness λ = 199. The column is subjected to the axial load N_{sd} = 180.4 kN, bending moment at the top $M_{sd,sup}$ = 17.92 kNm, bending moment at the bottom of the column $M_{sd,inf}$ = 4.45 kNm. The material is a C40 concrete (fck = 40 MPa), 16 reinforced steel bars ϕ = 16 mm, resulting in the associated N, M, 1/r diagram, Fig. 6.

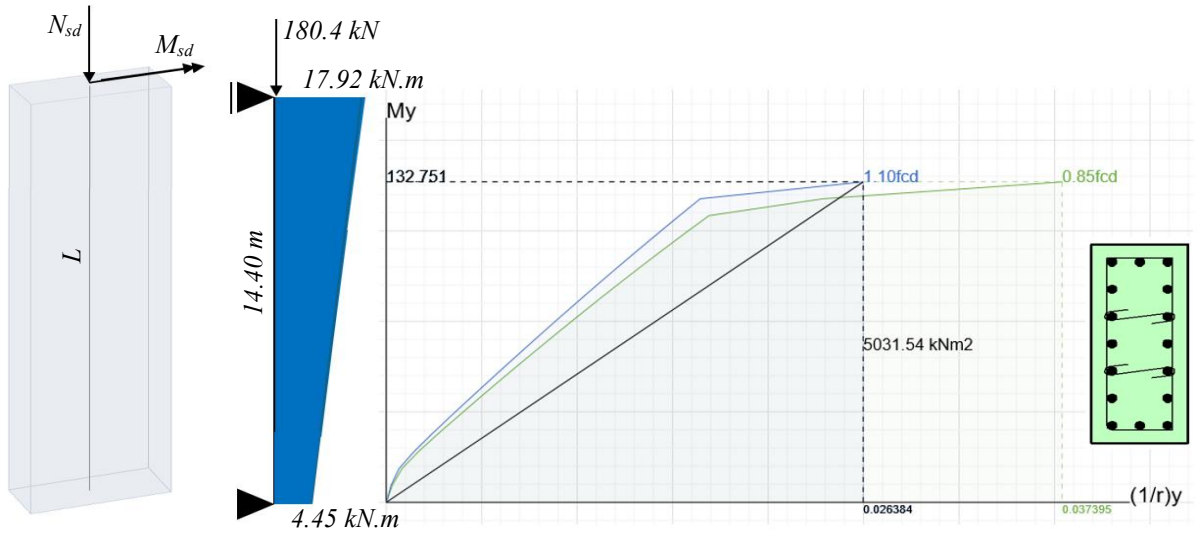


Figure 6. Reinforced concrete column

For a slenderness greater than 140, the general method should be used [1]. Thus, the TQS v.25 solves the problem considering a discretization of 10 elements with the secant stiffness EI_{sec} calculated from the N, M, 1/r diagram, and performing a complete geometric nonlinear analysis. Also, for this problem, it is possible to obtain the exact solution [6]:

$$M(x) = -EI\mu^2 \left\{ \left[-\frac{\alpha + \alpha cos(\mu L)}{sin(\mu L)} \right] sin(\mu x) + (\alpha) cos(\mu x) \right\}. \qquad (37)$$

The exact solution and the results provided by TQS v.25 are compared with the proposed element. It consists of a mixed formulation modeled with a single element considering the analytical interpolation functions [6], "MB1-Analytical", and are shown in Fig. 7. Also, for comparison purposes, the results obtained using the mixed formulation with Hermitian interpolation ("MB1-Hermitian"), as well as the conventional displacement-based elements with analytical ("DB1-Analytical") and Hermitian ("DB1-Hermitian") shape functions, are also presented in Fig. 7.

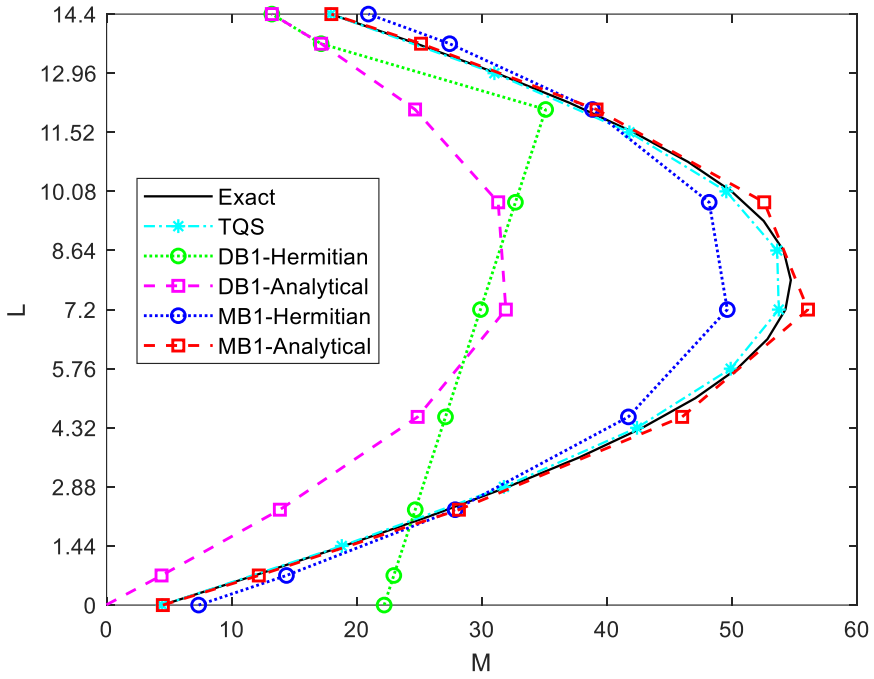


Figure 5. Bending moment distribution along the proposed reinforced column for different formulations

The bending moment distribution shows that the proposed formulation ("MB1-Analytical") [6] with a single element follows the results given by the exact solution and the TQS v.25 response with discretized column. Despite some interesting results, the mixed formulation based on Hermitian functions give unsafe results for the column design, requiring at least two elements to properly represent the column curvature. Elements that use displacement-based formulations are not capable of predicting the column behavior with one element. In such cases, a more refined with 3 or 4 elements are necessary.

4 Conclusions

A mixed beam element formulation using analytical interpolation functions, implemented in OpenSees, was used for the analysis of very slender columns modeled with a single element. The shape functions were calculated directly from the solution of the equilibrium differential equation of a deformed infinitesimal element, incorporating the influence of axial forces. The results clearly demonstrated the accuracy and applicability of the proposed element in predicting the behavior of slender columns using only one element, providing similar results to the general method proposed by the NBR 6118 [1]. Extensions of the formulation are being explored to consider the shear deformation and to reproduce the exact solution.

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