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Elastic-plastic mechanical properties of cross-shaped honeycomb materials using Timoshenko bending theory and Hill failure criterion

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Abstract. Honeycomb materials are characterized by a periodic structure composed of unit cells that exhibit interesting properties such as a high stiffness-to-weight ratio, functional versatility, and structural efficiency. Due to these characteristics, these materials are employed in various fields including architecture, engineering, and biomedicine. The mechanical properties of such materials are intrinsically influenced by the topology of the cells, which determines the overall response of the cellular medium. Therewith, the modeling and characterization of the mechanical behavior of cellular materials have intensified in recent decades, emphasizing nonlinear responses under different loading conditions. Numerical models are usually based on the theories of elasticity and plasticity. Therefore, the present study aims to investigate the mechanical properties of two-dimensional cellular materials, focusing on unit cells with cross-shaped topology. The proposed model uses planar frame members based on Timoshenko bending theory combined with a plasticity analysis. The classic Hill criterion characterizes the failure modes. For the analyzed problems, the global response was obtained through numerical homogenization. The results showed good agreement with the available experimental data.

Keywords: cross-shaped honeycomb materials; elastic-plastic analysis; Timoshenko bending theory; Hill failure criterion.

1 Introduction

Honeycomb materials, also known cellular materials, are so named because of their resemblance to the honeycombs found in beehives. The mechanical properties of these cellular materials depend primarily on the configurations of the unit cells that form the solid structure, rather than on the properties of the matrix materials. The matrix, in this context, is defined as the solid material that fills the spaces between the unit cells. This is because the geometry of the unit cells exhibits high performance in several mechanical characteristics, such as fracture toughness, impact resistance, heat dissipation, vibration damping, and others [1].

In civil engineering, materials such as honeycombs are widely used in structural reinforcement applications and sandwich configurations due to their remarkable properties, including high ductility, low density, precise thickness control, and good resistance to abrasion and fire [2]. The technical literature comprises numerous studies investigating the mechanical behavior of these materials under various loading conditions, both in-plane and out-of-plane, and involving either static loads or impact scenarios. One such study is that of Zhang et al. [3], in which the authors proposed a model based on continuum damage mechanics to analyze the behavior of sandwich panels with honeycomb cores. The results emphasize the need for further research, particularly focused on the formulation of constitutive laws and damage criteria that more accurately represent the behavior of these materials under diverse types of loading. Wang and McDowell [4] analyzed the mechanical properties of different in-plane

topologies. Their findings highlight the complexity of the behavior of honeycomb materials under specific loading conditions and underscore the need for further improvement and refinement in the formulations used to better understand their mechanical performance.

In light of the foregoing, the present study aims to investigate the mechanical properties of in-plane cellular materials, focusing on cross-shaped unit cell configurations. To achieve this goal, the methodological framework is based on the concepts of planar frame theory, which are grounded in elasticity and plasticity theory through Timoshenko's beam theory, further incorporating Hill's failure criterion to assess strength of the material.

2 Planar frame theory

In the present work, the notation introduced by Powell [5] is adopted, considering both small displacements and strains, with specific application to elastic planar frame elements. For this purpose, Timoshenko beam theory [6] is employed, enabling the consideration of shear deformation effects. In this model, the distortion induced by shear is modeled as an additional rotation of the cross-section.

Consider a plane frame consisting of m elements connected by n nodes, as shown in Figure 1(a). The global generalized displacements are defined by the nodal coordinates, which govern the spatial configuration of the structure. The shape-change matrix represents only the displacements and rotations associated with rigid-body motion and is not capable of representing the deformable behavior of the system. To describe the elastic behavior and shape modifications, additional kinematic variables are introduced, called generalized strains [7]. These strains are represented by the vector $\{\Phi_b\}$, which includes the relative rotations at the ends (Φ_i^b, Φ_j^b) and the axial strain δ_b, associated with the elongation of the element, as shown in Figure 1(b) [8].

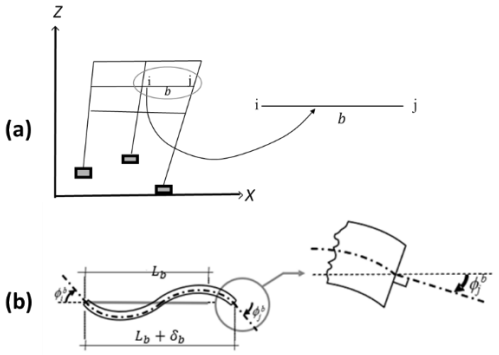


Figure 1. (a) Representation of element *b*; (b) Generalized deformations of element *b* [8]

The local displacements of the analyzed component can be obtained by rotating the coordinate axes, which is expressed by the product of the transformation matrix and the nodal displacement vector defined in the global coordinate system. The deformed configuration of a frame element is determined from the infinitesimal increments applied to its respective nodes. Specifically, relative rotations vary linearly as a function of the vertical and angular displacements imposed on the nodes, while axial deformation is directly related to the difference between the horizontal displacements of nodes *i* and *j*. The kinematic equation for frame elements, written in terms of the displacement matrix, is given by:

$$\{d\Phi\}_b \cong [B_E(U)]_b\{dU\}; \qquad \{\Phi\}_b \cong [B_E^0]_b\{U\} \tag{1}$$

where $[B_E(U)]_b = [B_E^0]_b$ is the expanded kinematic transformation matrix, constructed to simultaneously represent all elements within the global coordinate system. To achieve this, zero columns are inserted at positions that do not correspond to the degrees of freedom of the considered element:

$$[B_E^0]_b = \begin{bmatrix} 0 & 0 & 0 & \dots & \frac{sin\alpha_b}{L_b} & -\frac{cos\,\alpha_b}{L_b} & 1 & \dots & -\frac{sin\alpha_b}{L_b} & \frac{cos\,\alpha_b}{L_b} & 0 & \dots \\ 0 & 0 & 0 & \dots & \frac{sin\alpha_b}{L_b} & -\frac{cos\,\alpha_b}{L_b} & 0 & \dots & -\frac{sin\alpha_b}{L_b} & \frac{cos\,\alpha_b}{L_b} & 1 & \dots \\ 0 & 0 & 0 & \dots & -cos\,\alpha_b & -sin\alpha_b & 0 & \dots & cos\,\alpha_b & sin\alpha_b & 0 & \dots \end{bmatrix} \tag{2}$$

The static variables in frame structures are divided into internal and external forces, which are related through the equilibrium equations. The static equilibrium of a structure is established based on the principle of virtual work, by equating internal and external virtual work for arbitrary virtual displacements. Therefore, the formulation

of the global equilibrium equations is given by:

$$\sum_{b=1}^{m} [B_E^0]_b^t \{M\}_b = \{P\} \tag{3}$$

where $[B_E^0]_b^t$ is the kinematic transformation matrix of the element, and $\{M\}_b$ is the generalized stress matrices. {P} represents the components of forces or moments distributed along the structure or concentrated at specific points of the elements. Distributed loads are usually converted into equivalent nodal forces, concentrated at the nodes of the elements. These forces are assembled into a vector of generalized nodal forces, as shown in Equation (4) [7,8].

$$\{P\}^t = \{p_{ui}, p_{wi}, p_{\theta i}, \dots p_{uj}, p_{wj}, p_{\theta j} \dots, p_{un}, p_{wn}, p_{\theta n}\} \tag{4}$$

The relationship between stress and strain matrices is defined by the constitutive laws of the material, which describe the mechanical behavior of structural elements under loading. According to Perdomo et al. [9], the relationship between the generalized strain matrices, $\{\Phi^e\}_b$, of element can be expressed in matrix form, as shown in Equation (5):

$$\{\Phi^e\}_b = [F_0]_b \{M\}_b \tag{5}$$

where $[F_0]_b$ is the elastic flexibility matrix, derived from classical Timoshenko beam theory, as detailed below:

$$[F_0]_b = \begin{bmatrix} \frac{L_b}{3EI} & -\frac{L_b}{6EI} & 0 \\ -\frac{L_b}{6EI} & \frac{L_b}{3EI} & 0 \\ 0 & 0 & \frac{L_b}{EA} \end{bmatrix} + \begin{bmatrix} \frac{1}{k_s GAL_b} & \frac{1}{k_s GAL_b} & 0 \\ \frac{1}{k_s GAL_b} & \frac{1}{k_s GAL_b} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

2.1 Homogenization of elastic-plastic frames

Plasticity theory aims to describe the behavior of materials that exhibit both elastic and plastic strains. Materials that respond in this way are referred to as elastoplastic. This theory provides mathematical models governed by the laws of elasticity, hardening rules, and failure criteria that approximate the mechanical behavior of materials.

Failure criteria are employed to identify the stress limit beyond which the material can no longer sustain the applied loading conditions without experiencing structural failure. Among the various criteria proposed in the literature, this study adopts the in-plane quadratic Hill criterion as the basis for evaluating the strength and stability of honeycomb structures, whose mechanical response is characterized by pronounced anisotropy due to varying loading directions. The choice of this criterion is justified by its ability to appropriately represent the effects of anisotropy under multiaxial loading conditions, making it a suitable analytical tool for investigating the behavior of these materials. The quadratic Hill yield function [10] is defined as:

$$Y^2 = \sigma_1^2 + \left(\frac{2R_0}{1+R_0}\right)\sigma_1\sigma_2 + \left(\frac{1+R_{90}}{1+R_0}\right)\frac{R_0}{R_{90}}\sigma_2^2 \quad \forall\, \sigma_3 = 0 \tag{7}$$

where σ_1 e σ_2 are the principal values of the stress tensor in each direction of application, R_0 e R_{90} are the ratio of plastic strains along the width and thickness, also called the Lankford coefficients.

3 Results and Discussions

To investigate the elastic properties of the unit cell with cross topology using the planar frame formulation, a rectangular cross-section with a base of 1 mm and a height of 0.15 mm was adopted. For the modeling, in addition to these geometric parameters, specific mechanical properties were considered, assuming the use of A36 steel as the material. The length of each bar in the unit cell was set to 0.5 mm, the modulus of elasticity (E) was taken as 207 GPa, and Poisson's ratio (υ) was set to 0.3. Additionally, the yield stress (σ_0) was defined as 248 MPa, and the shear correction factor (k_s) was assumed to be 5/6.

Through the application of these parameters, combined with frame theory and the boundary conditions illustrated in Figure 2, it was possible to determine the elastic constants of honeycomb materials subjected to

horizontal and vertical tensile loading and shear, as functions of the unit cell's geometric parameters (base and height), the material's elastic modulus, and the element length, as presented in Table 1.

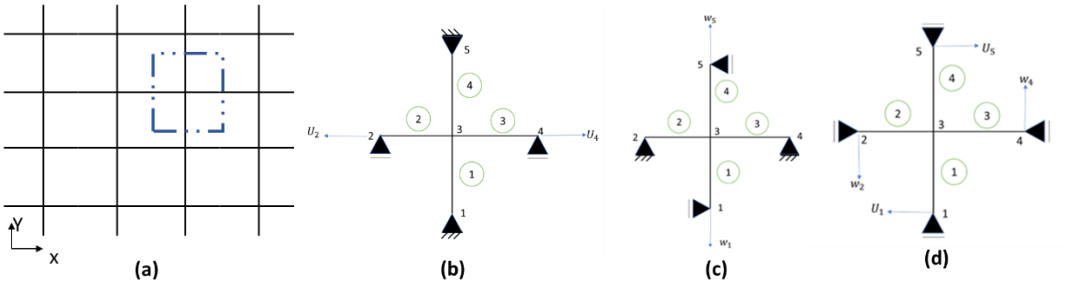


Figure 2. (a) Supercell; (b) boundary condition for horizontal tension; (c) boundary condition for vertical tension; (d) boundary condition for shear.

Table 1. Elastic property values of the cross-shaped unit cell

Requests	Tensile horizontal	Tensile vertical	Shear
Elastic constants in the direction *x*	$C_{11} = \frac{bhE}{L}$	$C_{12} = 0$	$C_{13} = 0$
Elastic constants in the direction *y*	$C_{12} = 0$	$C_{22} = \frac{bhE}{L}$	$C_{23} = 0$
Elastic constants in the direction *xy*	$C_{13} = 0$	$C_{23} = 0$	$C_{33} = \frac{3bhE^3}{4(h^2+12)L^3}$

The analyzed material exhibits anisotropic mechanical behavior, as shown in Table 1. It is noteworthy the results obtained for the cross-shaped configuration under horizontal and vertical tensile loading are consistent with those reported by Wang and McDowell [4], thereby supporting the validity and relevance of the findings presented in this study. Regarding the shear formulation, the value differs from that reported by Wang and McDowell [4] due to the adoption of different theoretical approaches. While the authors employed the Euler–Bernoulli beam theory in their study, the present research uses the Timoshenko beam theory, which accounts for shear deformation effects.

After determining the elastic constants, the results were validated by comparing the forces and displacements obtained from calculations based on the presented formulation and performed using computer algebra software with those obtained from finite element numerical simulations carried out in Abaqus CAE. The results are presented in Table 2.

Table 2. Comparison of forces and displacement of the Cross Unit Cell

Requests	Manual calculation		Finite element calculation	
	Forces (kN)	Displacement (mm)	Forces (kN)	Displacement (mm)
Tensile horizontal	$P_{U2} = -31.05$	$U_2 = 0.5$	$P_{U2} = -31.05$	$U_2 = 0.5$
Tensile vertical	$P_{v1} = -31.05$	$w_1 = 0.5$	$P_{v1} = -31.05$	$w_1 = 0.5$
Shear	$P_{u1} = 0.6927$ $P_{v2} = 0.6927$	$U_1 = 0.25$ $w_2 = 0.25$	$P_{v2} = 0.6872$ $P_{u3} = 0.6872$	$U_1 = 0.25$ $w_2 = 0.25$

Upon analyzing the results presented in Table 2, it is evident that the discrepancies between the values obtained from the two software tools are negligible, indicating remarkable consistency in the calculations, enhancing the reliability of the results, and validating the adopted methodological approach. It is noteworthy that, when the material is subjected to shear stress, it presents a relative error of 0.8%. The robustness of the methodology is further supported by the comparison of the resulting forces between the supercell and the homogenized structure, as shown in Table 3.

The results presented in Table 3 confirm the consistency between the supercell and the homogenized structure, with minimal relative error. The maximum relative error was 3.94%. In order to validate the methodology adopted to describe the mechanical behavior of honeycomb materials, three stainless steel meshes, referred to as SSWM, were analyzed. These meshes are composed of unit cells with a cross configuration and a circular cross-section. Each cell was strategically the cells were strategically positioned along the axis of symmetry of the circular cross-section of the mesh, as shown in Figure 3. It is worth noting that this procedure was also applied to the other meshes, resulting in adjustments to the individual lengths of each bar in the unit cell. The new

lengths obtained from the optimization for each mesh are: 0.4135 mm, 0.3075 mm, and 0.2385 mm for the 30×32, 40×32, and 50×34 meshes, respectively.

Table 3. Comparison of the resultant forces of the cross-shaped unit cell

Requests	Quantities of unit cells	Force (kN) Supercell	Force (kN) Homogenized Structure	Relative error (%)
Tensile horizontal	3 x 3	31.05	31.05	0%
	7 x 7	31.05	31.05	0%
	7 x 8	31.05	31.05	0%
	20 x 20	31.05	31.05	0%
Tensile vertical	3 x 3	31.05	31.05	0%
	7 x 7	31.05	31.05	0%
	7 x 8	31.05	31.05	0%
	20 x 20	31.05	31.05	0%
Shear	3 x 3	3.13383	3.26236	3.94%
	7 x 7	3.75545	3.77862	0.61%
	7 x 8	3.26079	3.2741	0.41%
	20 x 20	3.25409	3.26087	0.20%

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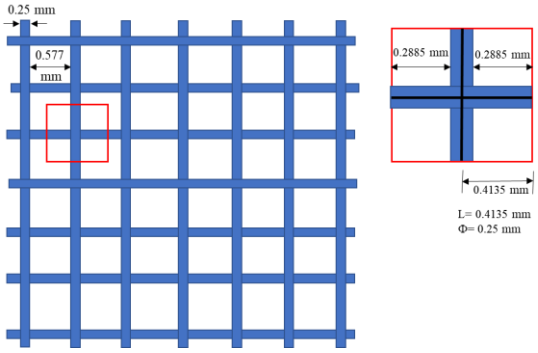


Figure 3. Positioning of the unit cell along the symmetry axis of the 30x32 mesh.

Furthermore, the mechanical parameters used in the modeling of the SSWM meshes were defined according to the configuration of each mesh. The 30x32 and 40x32 meshes have a diameter of 0.25 mm, while the 50x34 SSWM mesh has a diameter of 0.23 mm. For all meshes, a Poisson's ratio of 0.3 and a modulus of elasticity of 151 GPa were considered. Regarding the yield stress, values of 306.1224 MPa were adopted for the 30x32 and 40x32 SSWM meshes, and 216.3415 MPa for the 50x34 SSWM mesh [11]. It is worth noting that the adopted shear correction factor was 9/10, as established in the literature

From the application of these parameters, using plane frame theory and adopting the boundary conditions shown in Figure 3, it was possible to determine the elastic constants of the unit cell, whose values are presented in Table 4.

Table 4. Values of the elastic constants of the Cross-unit cell

Requests	Tensile horizontal	Tensile vertical	Shear
Elastic constants in the direction *x*	$C_{11} = \frac{E\pi\Phi^2}{8L}$	$C_{12} = 0$	$C_{13} = 0$
Elastic constants in the direction *y*	$C_{12} = 0$	$C_{22} = \frac{E\pi\Phi^2}{8L}$	$C_{23} = 0$
Elastic constants in the direction *xy*	$C_{13} = 0$	$C_{23} = 0$	$C_{33} = \frac{15E\pi\Phi^4}{16(27\upsilon\Phi^2 + 27\Phi^2 + 80)L^3}$

Subsequently, the yield criterion proposed by Hill [10] was applied to different configurations of the cellular structure (SSWM) to evaluate the failure resistance of materials with cross-shaped topology. The results obtained from the plasticity analysis show that the structure reached yield stress, followed by ultimate stress. Under these conditions, no discernible difference exists between the current state and the failure state of the structure, as illustrated in Figure 4.

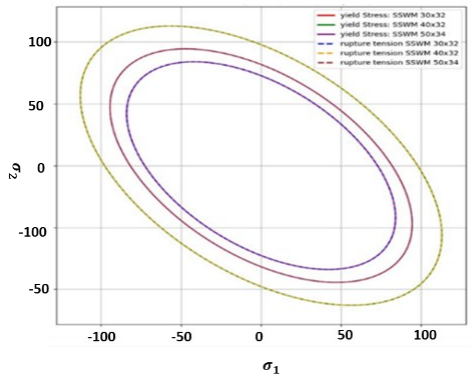


Figure 4. Failure criterion

From the results obtained in the plastic analysis, conducted both using computer algebra software and within the Abaqus/CAE simulation environment, it was possible to determine the curve representing the relationship between the applied load and the resulting displacement under horizontal tension. In these simulations, the CPS4 finite element (4-node bilinear plane stress quadrilateral) was employed. To construct this curve, a homogenized structure with a fixed base and a vertically distributed load applied at the top was considered (Figure 5(a)). The dimensions of the homogenized structure were adopted according to the parameters established in the study by Patel, Joshi, and Makawana [11], in which the authors performed tensile tests on three specimens for each type of SSWM, with a width of 100 mm and a total length of 500 mm. For these tests, steel plates, 100 mm in length, were added at both ends, resulting in an effective test length of 300 mm (Figure 5(b)).

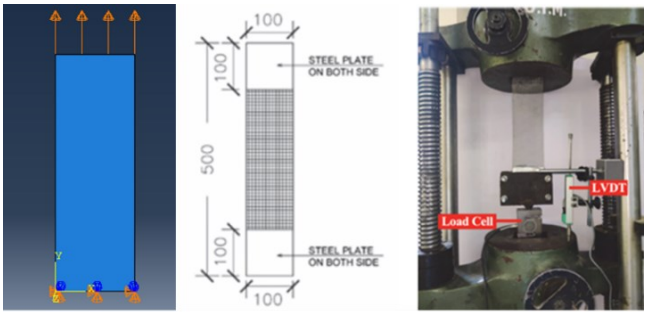


Figure 5. (a)Homogenized structure; (b) SSWM tensile test performed by Patel, Joshi, and Makawana [11]

The results obtained from the simulations were subsequently compared with the experimental data provided by Patel, Joshi, and Makawana [11], as shown in Figure 6.

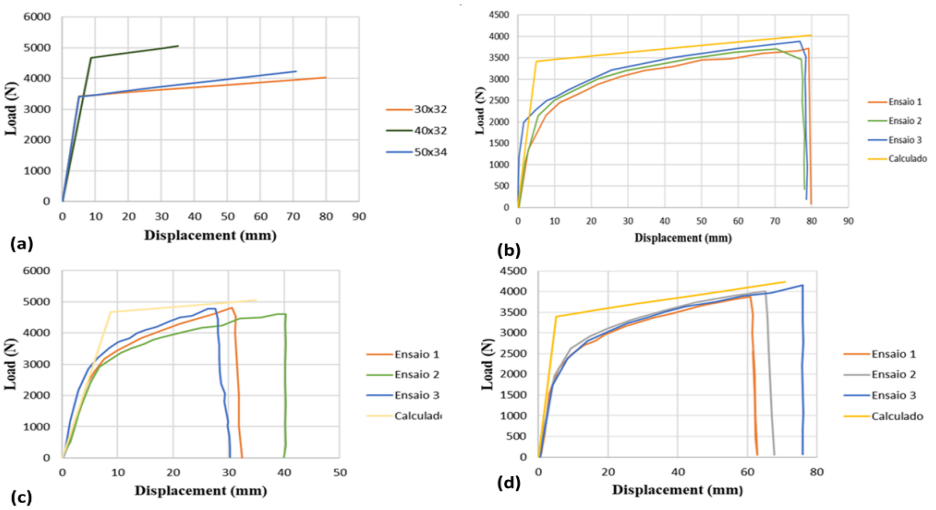


Figure 6. Load vs. displacement: (a) all meshes simulated in the software; (b) SSWM 30x32; (c) SSWM 40x32; (d) SSWM 50x34

The methodology adopted in this study is capable of representing, with good approximation, the mechanical behavior of the supercell. Overall, the results reinforce the robustness of the proposed approach for characterizing the mechanical properties of materials with cross-shaped topologies, demonstrating the effectiveness of using accessible and low-cost computational tools in the modeling and analysis of honeycomb structures.

4 Conclusions

The results obtained demonstrate that the adopted methodology satisfactorily and accurately describes the behavior of honeycomb materials in the unit cell cross configuration under various loading conditions. Furthermore, they highlight the relevance of the proposed model, which precisely characterizes the transition in the elastic behavior of honeycomb materials within this topology.

It is noteworthy, however, that there is a broad area of research to expand knowledge about the mechanical properties of these materials. Based on this, the following suggestions for future research are proposed: explore other unit cell configurations to evaluate how geometry influences the mechanical properties of honeycomb materials; investigate the feasibility of applying the plane frame methodology with different failure criteria; and analyze honeycomb materials under varying loading conditions under dynamic loads.

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