A Comparison Between the Methods of Finite Volume and the Boundary Element for Solving Homogeneous and Piecewise Two-Dimensional Media Wave Problems

Thiago G. Balista[1,a], Carlos F. Loefffler[1,b], Julio T. A. Chacaltana[2,c], Renan A. Peres[2,d], Luciano O. C. Lara[1,e]
[1]*Mechanical Engineering Post-Graduate Program, Federal University of Espírito Santo.*
[2]*Environmental Engineering Post-Graduate Program, Federal University of Espírito Santo.*
[1,2]*Av. Fernando Ferrari, 540-Bairro Goiabeiras, 29075-910 Vitoria, ES, Brazil*
[a)]*thiago.balista@gmail.com,* [b)]*loefflercarlos@gmail.com,* [c)]*juliotac2019@gmail.com,* [d)]*renan.peres@edu.ufes.br*
[e)]*castrolara@hotmail.com*

Abstract. This work presents a comparison between the boundary element method with direct interpolation and the finite volume method, the latter in an unstructured version, solving problems of scalar wave propagation in homogeneous and heterogeneous media in two dimensions. The objective is to evaluate the capability of the two numerical techniques in solving dynamic problems of this nature, which are known to be challenging in terms of accuracy of results. Evaluations regarding the convergence rate as a function of the mesh refinement, the numerical robustness involving the stability of the temporal advance schemes, among other particularities, are carried out. These evaluations are performed by solving a classical problem using both methods, allowing a direct comparison of the accuracy of the obtained results.

Keywords: Boundary element, direct interpolation method, finite volume method, homogeneous and heterogeneous media, dynamic problems

1 Introduction

It is a proven fact that, during the propagation of a pressure wave in a fluid medium, the particles in the medium undergo small displacements, subsequently returning to their initial position. It is also verified in practice that the velocity of these fluid particles is much lower than the rotational velocity of the pressure wave. This data, together with the assumption of an isentropic process during the propagation of a pressure wave in a compressible fluid medium, as well as the principles of conservation of mass and momentum, make it possible to eliminate the terms involving velocity and obtain the partial second-order differential equation for pressure, namely the Acoustic Wave Equation [1], which takes into account the inhomogeneity of the continuous medium:

$$\left[\frac{c^2}{K(X)}u(X,t)_{,i}\right]_{,i} = \frac{1}{K(X)}\ddot{u}(X,t) \tag{1}$$

In this solution, $u(X,t)$ represents the displacement caused by the pressure wave in the fluid medium, and c is the speed at which the pressure wave propagates in the fluid medium. It is worth noting the relationship $K(x) = \rho c2$, where $E(X)$ is the volumetric detailed modulus and ρ is the specific mass, a notation commonly found in seismic exploration studies.

It is worth noting that there is an undeniable similarity between the Equation of Acoustics for fluid media and the problem of propagation of expansion waves in solid media. It is known that delay waves, because they travel more slowly than expansion waves, can be disregarded explicitly in typical seismic analysis procedures. Thus, the pressure $u(X,t)$ can specifically represent the acoustic wave traveling in a rocky solid medium, typically heterogeneous, or simply describe the transverse motion in a metallic membrane, also fluid or heterogeneous.

A partial differential equation as expressed by eq. (1) does not have analytical solutions, especially if the medium is inhomogeneous. Therefore, it is imperative to employ approximate discrete solution methods, notably

the Finite Element Method (FEM), the Finite Difference Method (FDM), the Finite Volume Method (FVM), and the Boundary Element Method (BEM). Despite their many similarities, the first three are considered domain techniques, in which both discretization and numerical integration are performed over the physical domain. In the BEM, such integrations were restricted to the boundary.

There are some problems, notably those in which the medium is heterogeneous, where the variation in properties across the domain is immediately computed by domain methods, being included directly in elements, volumes, or cells. However, these variable properties require the use of special strategies in the case of the BEM. The fact of sectoral or gradual heterogeneity is also quite crucial for the mathematical model to be adopted by the BEM. In this work, only homogeneous piecewise cases are solved, and in these, eq. (1) can be simplified to be expressed as:

$$u(X,t)_{,ii} = \frac{1}{c^2}\ddot{u}(X,t) = \frac{\rho}{K(X)}\ddot{u}(X,t) \tag{2}$$

Therefore, our piecewise problems like properties are computed directly on finite volumes, while in the Boundary Element Method, techniques like subregions or superposition of domains need to be employed.

2 Finite Volume Method

The Finite Volume Method (FVM) is a numerical technique used to solve partial differential equations. It is based on discretizing the domain into small volumes and ensuring the balance of properties such as mass, energy, and momentum in each volume. Initially, the diffusion discovery is integrated into a generic finite volume $\Omega(X)$ so that:

$$\int_{\Omega} [K(X)u(X,t)_{,i}]_{,i} d\Omega = \int_{\Omega} \rho\ddot{u}(X,t)\, d\Omega \tag{3}$$

In the methodology employed here, the right-hand side of eq. (1) is first solved in isolation from the dynamic part, without any loss of generality. This is because the time-dependent term is easily operationalized through incremental feedforward schemes, also known as direct integration techniques, which have also worked in the main known discrete methods.

Considering the left-hand side of eq. (2), the focus is on developing a numerical solution in non-finite volumes, capable of efficiently representing irregular boundaries and enabling differentiated refinement in some areas of the domain, if necessary. Thus, the numerical model is based on a base triangle, which must be related to neighboring triangles and the unitary external normal to the edges of the base triangle, as illustrated in Fig. 1

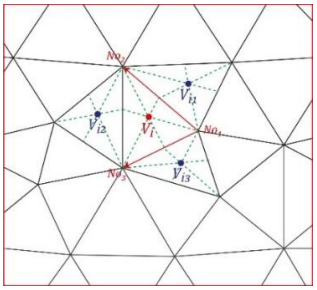


Figure 1. Basis triangular finite volume (V_i) and its neighbor volumes.

Using the Divergence Theorem [2], the domain integral in the base triangle can be rewritten in terms of boundary integrals and be decomposed into the sum of three integrals involving the faces of the triangular volume, according to the following relation:

$$\begin{aligned}\int_{\Omega} [K(X)u(X,t)_{,i}]_{,i} d\Omega &= \int_{\Gamma} [K(X)u(X,t)_{,i}\, n_i] d\Gamma \\ &= \int_{\Gamma_1} [K_1 u_1(X,t)_{,i}\, n^1{}_i] d\Gamma_1 + \int_{\Gamma_2} [K_2 u_2(X,t)_{,i}\, n^2{}_i] d\Gamma_2 \\ &\quad + \int_{\Gamma_3} [K_3 u_3(X,t)_{,i}\, n^3{}_i] d\Gamma_3\end{aligned} \tag{4}$$

In eq. (4), Γ_1 is the surface of the face joining nodes 1 and 2, Γ_2 is the surface of the face joining nodes 2

and 3, and Γ_3 is the surface of the face joining nodes 3 and 1. The unitary normal vectors on each edge are given by ni. Each integral on the right-hand side of Eq. (4) can be approximated using the mean value theorem for integrals, which results in the following numerical approximation:

$$\int_{\Gamma} [K(X)u(X,t)_{,i}\, n_i] d\Gamma = K_1 u_1(X,t)_{,i}\, n^1{}_i L_1 + K_2 u_2(X,t)_{,i}\, n^2{}_i L_2 + K_3 u_3(X,t)_{,i}\, n^3{}_i L_3 \tag{5}$$

In Eq. (3), L_1, L_2, and L_3 are the lengths of the edges between adjacent nodes. The calculation of each of the gradients that appear in Eq. (5) is performed by applying the Gradient Theorem [3]. After some algebraic manipulation, it can be rewritten as in the discrete form commonly used in the finite volume method, after substituting the numerical approximations of the calculated gradients:

$$\int_{\Gamma} [K(X)u(X,t)_{,i}\, n_i] d\Gamma = A_1 u_1 + A_{i1} u_{i1} + A_{i1} u_{12} + A_{i2} u_{i2} + A_{i3} u_{i3} \tag{6}$$

The eq. (6) is used to construct the system of equations that represents the discrete form of the continuous domain of interest. In Eq. (6), the discrete value of the function $u(X,t)$ is calculated at the volume's centroid and, based on information from neighboring volumes, thus allowing the elements to be compatible and the system of equations to be constructed for different finite volumes [4, 5]. Assume that the values of $u(X,t)$ at the vertices of each triangular element are known, based on the interpolation of the values of $u(X,t)$ from the centroid to the nodes. The next step consists of integrating the acoustic term of the Wave Equation into the finite base volume Ω_i, as shown in the following proposal:

$$\int_{\Omega_i} \rho \ddot{u}(X,t)\, d\Omega = \frac{\Omega_i}{K_i} \left[\frac{u_i^{t+1} - 2u_i^t + u_i^{t-1}}{\Delta t^2}\right] \tag{7}$$

In eq. (5), Ω_i is the volume of the finite base volume and t is the discrete time. Without loss of generality, considering the base volume of triangular cross-section and unit thickness, the value associated with this volume is the area of the triangular mesh element itself. In eq. (7), the finite difference approximation of the second time derivative of the primal variable $u(X,t)$ with second-order approximation in time was used. In this way, an explicit integration scheme can be set up in which one advances incrementally over time by choosing an appropriate integration step Δt [6, 7] and starting from prescribed initial and boundary conditions. Then, the explicit scheme will provide the pressure value in the future $t+1$, provided that the pressure values are known in the present t and in the past $t-1$, throughout the domain of interest.

3 Boundary Element Method

Following typical Boundary Element Method (BEM) procedures, the differential government proposal must be expressed in integral form, using an auxiliary function that consists of the description of the fundamental solution, which can be a Green's function directly associated with the problem in question or a simplified form of the same, relative to a stationary problem. Taking advantage of the latter possibility, a fundamental solution of the Poisson problem $u^*(\xi;X)$ [8] is employed, and thus one can resort to simple formulations, which can employ discrete-time advance methods, similar to that shown on the right-hand side of Eq. (7). Therefore, considering that the non-homogeneity is sectoral, the strong integral form related to the governing equation of the acoustic wave can be rewritten as according to the Domain Superposition Technique (DST) [9].

$$\begin{aligned} K^e \int_{\Omega^e} u_{,ii}\,(X) u^*(\xi;X) d\Omega^e + K^s \int_{\Omega^i} u_{,ii}\,(X)^i u^*(\xi;X) d\Omega^i = \rho^e \int_{\Omega^e} \ddot{u}(X,t)\, u^*(\xi;X) d\Omega^e + \\ \rho^s \int_{\Omega^i} \ddot{u}(X,t)^i\, u^*(\xi;X) d\Omega^i \end{aligned} \tag{8}$$

In the DST, the entire problem is modeled by a superposition of a surrounding homogeneous domain along with complementary internal subdomains, which have distinct properties (see Fig. 2). Due to space limitations, more details can be found in the specialized literature [9-11]. Therefore, in eq. (8), K^e and ρ^e represent, respectively, the stiffness and inertia properties of the surrounding domain, while K^i and ρ^i correspond to the same properties in the internal domain. It is also possible to consider that $K^s(X) = K^i(X) - K^e(X)$ and $\rho^s(X) = \rho^i(X) - \rho^e(X)$. The fundamental solution $u^*(\xi;X)$ is stationary, corresponding to the solution of a stationary problem with an infinite domain, subject to a concentrated source, applied at a specific point in the spatial domain X, called the source point ξ. The expression for the fundamental solution can be found in various references [12].

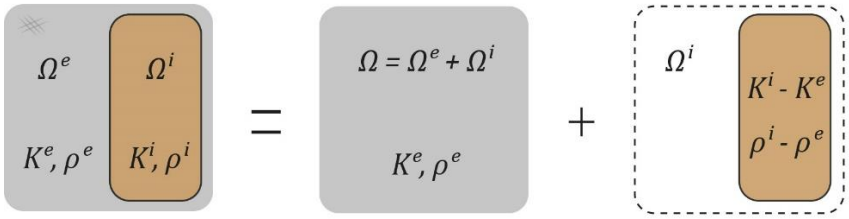


Figure 2. Complete and sectorial domains with homogeneous properties.

It can be observed that the left-hand side of eq. (8) consists of the Laplacian operator, which is a self-adjoint differential operator. Under these conditions, the application of integration-by-parts procedures and the Divergence Theorem leads to an inverse form associated with the Laplacian operator, such that for a given sector Ω_e one has:

$$K^e \int_{\Omega^e} u_{,ii}(X) u^*(\xi;X) d\Omega^e = -K^e \left[c(\xi)u(\xi) - \int_{\Gamma^e} q(X)u^*(\xi;X)\, d\Gamma^e + \int_{\Gamma^e} u(X) q^*(\xi;X)\, d\Gamma^e \right] \tag{9}$$

It is worth noting that the same procedure can be applied in a similar way to the term involving Ω_i that the left-hand. In eq. (9), $u(X,t)$ is the scalar potential and $q(X,t)$ is its normal derivative; $u^*(\xi;X)$ is the fundamental solution and $q^*(\xi;X)$ is its normal derivative. Thus, the entire diffusive term here is now rewritten in terms of boundary integrals. However, the term related to the acoustic inertia needs to be similarly discretized. In the case of the DIBEM technique work, the complete kernel of the domain integral is interpolated by radial basis functions directly, according to eq. (10):

$$\ddot{u}(X,t) u^*(\xi;X) \cong {}^{\xi}\ddot{\beta}^j F^j(X^j;X) \tag{10}$$

The functions $F^j(X^j;X)$ are interpolation functions and belong to the class of radial functions [13]. Several classes of radial functions can be used, but among the most robust is the thin-plate radial basis function, which is used here in all simulations. After the discretization process, these X points generate the nodal points at which the potential $u(X,t)$ is calculated. For each source point ξ, the interpolation given by Eq. (9) corresponds to a sweep of all base points X^j relative to the points X of the domain, weighted by the coefficients ${}^{\xi}\ddot{\beta}^j$. These coefficients can be obtained after discretization by solving a simple system of algebraic equations [14, 15]. To avoid singularities due to coincidences between the source and field points, the following should be considered:

$$[\ddot{u}(X) - \ddot{u}(\xi)] u^*(\xi;X) \cong {}^{\xi}\ddot{\beta}^j F^j(X^j;X) \tag{11}$$

Thus, the procedure is applied to the two domain terms in eq. (9) by adding two self-compensating integrals. Using primitives of the original radial function, the integral of the acoustic inertia term becomes:

$$\begin{gathered} \int_{\Omega} [\ddot{u}(X) - \ddot{u}(\xi)] u^*(\xi;X) d\Omega \approx {}^{\xi}\ddot{\beta}^j \int_{\Omega} F^j(X^j;X)\, d\Omega = {}^{\xi}\ddot{\beta}^j \int_{\Omega} \Psi^j_{,ii}(X^j;X)\, d\Omega = \\ {}^{\xi}\ddot{\beta}^j \int_{\Gamma} \Psi^j_{,i}(X^j;X) n_i(X)\, d\Gamma = {}^{\xi}\ddot{\beta}^j \int_{\Gamma} \eta^j(X^j;X) d\Gamma \end{gathered} \tag{12}$$

There is also the surplus term, which is immediately conveyed to the boundary through the Galerkin Tensor:

$$\ddot{u}(\xi) \int_{\Omega} u^*(\xi;X)\, d\Omega = \ddot{u}(\xi) \int_{\Omega} G^*_{,ii}(\xi;X) = \ddot{u}(\xi) \int_{\Gamma} G^*_{,i}(\xi;X) n_i\, d\Gamma \tag{13}$$

The coefficients β in eq. (12) are unknowns, being necessary to rewrite it as a function of the potential $\ddot{u}(X,t)$, using the following equation:

$$\left[{}^{\xi}\ddot{\beta} \right] = [F]^{-1} \left[{}^{\xi}\Lambda \right] [F] \{\ddot{\beta}\} = [F]^{-1} \left[{}^{\xi}\Lambda \right] \{\ddot{u}\} \tag{14}$$

In eq. (14) $[\Lambda]$ is a diagonal matrix composed of the values of the fundamental solution. The special discretization procedure that follows is relatively extensive and can be found in several references [16]. Once the boundary integral, which forms the basis of the BEM philosophy, is defined, the system formed by all the resulting equations, in its matrix form, will be given by:

$$\begin{bmatrix} H_{cc} & H_{ci} \\ H_{ic} & H_{ii} \end{bmatrix} \begin{Bmatrix} u_c \\ u_i \end{Bmatrix} - \begin{bmatrix} G_{cc} & 0 \\ G_{ic} & 0 \end{bmatrix} \begin{Bmatrix} q_c \\ q_i \end{Bmatrix} = - \begin{bmatrix} M_{cc} & M_{ci} \\ M_{ic} & M_{ii} \end{bmatrix} \begin{Bmatrix} \ddot{u}_c \\ \ddot{u}_i \end{Bmatrix} \tag{15}$$

Again, in eq. (15), further details about each term can be found in the specialized literature [9-11]. Thus, the temporal discretization is done by the Houbolt scheme, a variant of the finite difference technique, but specifically suited to the use of DIBEM.

4 Numerical Example

As mentioned, the governing scalar wave equation can represent an acoustic propagation of waves in a fluid or the movement in a solid, despite a slender structure, as a membrane. To better understand the physical problem, this example is described as comprised of a square membrane with unitary dimensions that is completely fixed on the boundary. Centered in the membrane exists a sub-region with different stiffness, constituting the sectorial homogeneity of the system. The initial conditions in velocities and displacements are null. The movement is induced by an external domain source, distributed uniformly over the central domain. The mathematical treatment of this term was not shown in the text for the sake of space, but both the Finite Volume Method and DIBEM solve it easily, generating a simple matrix column term. Fig. 3 illustrates the physical problem.

Despite the simplicity of this problem, the FVM non-structured code needs to be tested in this regular problem, since the algorithm elaborated to compute the influence of adjacent triangles is not elementary. Beyond this, the dynamic problems require special care concerning the stable conditions to choose the time steps, which are confirmed by FVM preliminary tests.

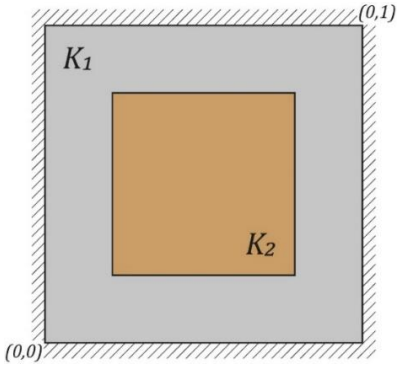


Figure 3. Square membrane with a central sector of different stiffness

Initially, the results are compared for $K_1 = K_2$ equal to unity. The results of the displacement at the central point of the membrane obtained with the finite volume method are presented in Fig. 4 below, for a mesh with 60 divisions on each edge and a time step equal to 0.05.

The results obtained with DIBEM for a similar mesh, with 320 linear boundary elements and 4 internal interpolating points, using an integration step equal to 0.019 seconds are shown in the Fig. 5.

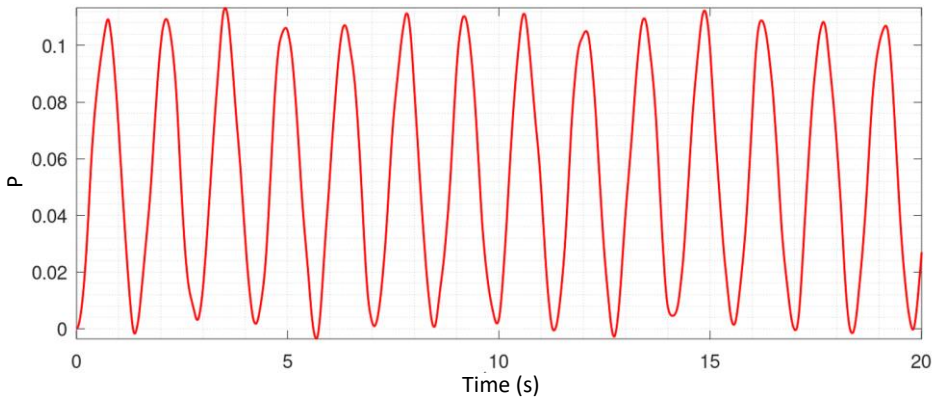


Figure 4. Time response at the central point of the homogeneous membrane with the FVM.

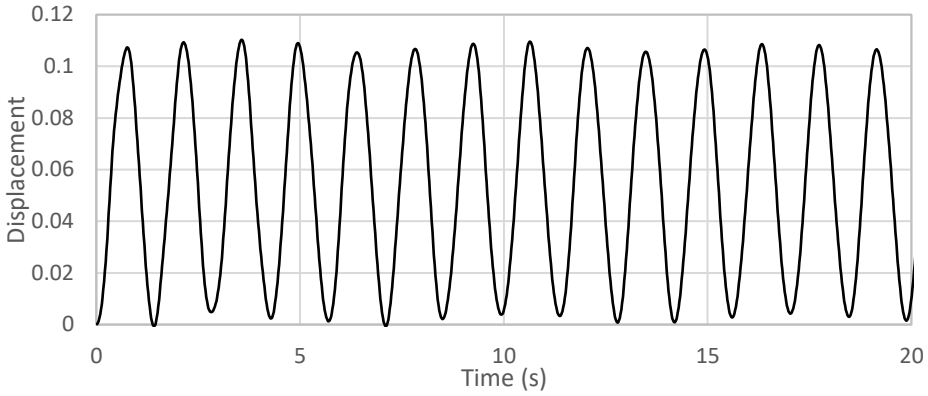


Figure 5. Time response at the central point of the homogeneous membrane with the DIBEM.

As can be noted, the results for both methods are very similar, with slight differences in amplitude relative to certain response peaks. The agreement could improve with mesh refinement, as the meshes used were very simple for both methods. Although there is no analytical solution for comparison, the expected peak and valley values for the dynamic response at the central point were confirmed (1s). Considering the wave propagation velocity as unitary, because stiffness and specific mass are also unitary, the time corresponding to the first peak describes the time it takes for the wave leaving the central point to reflect off the boundary and return to the point of origin.

In the second test, the central sector stiffness is doubled. The expectation is that the displacements in the central point in the membrane will be reduced and the initial wave reflection time will be shortened, due to the increased wave speed and propagation. Fig. 6 shows both the responses for the central point, using FVM and DIBEM, respectively. Meshes and integration time steps are the same used previously. The results were once again quite consistent with the physical expectations, since the maximum displacements were reduced and also the interval between the peaks. The agreement in results of both methods is evident.

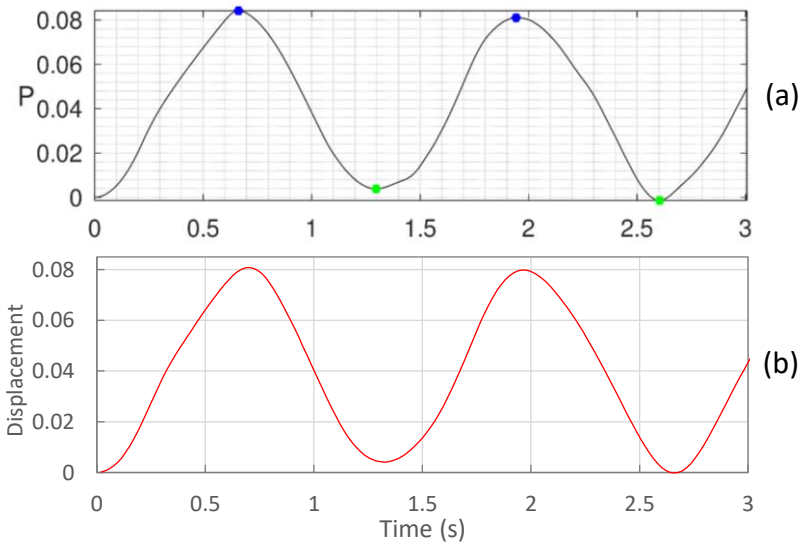


Figure 6. Comparison of time response at the central point of the sectorial membrane with $K_1 = 1$ and $K_2 = 2$ between (a) FVM and (b) DIBEM.

Regarding the DIBEM model, the accomplishment between the radial basis approximations with the Superposition Technique, used to compute the different properties, was considered a success, due to the agreement with FDM results. The superposition technique is an effective alternative to the well-known sub-region's technique. It is more accurate, due to the greater interactivity between different sectors, unlike the sub-

regions, which divide the domain in a similar sense as the Finite Element Method. This accomplishment was previously used only in Helmholtz problems, calculating frequencies for two-and three-dimensional problems. Still, it is used for the first time here in piecewise dynamic response problems.

5 Conclusions

This work has two main objectives. First, to verify the correctness of the algorithm developed for developing an unstructured two-dimensional finite volume model capable of representing non-regular geometries and also adapting to situations that preclude densifying the discretization in larger regions of property gradients or even the state variables of a given problem. The classical FVM model is relatively simple to implement, despite the variations that can be identified in the specialized literature, but the algorithmic scheme capable of modeling irregular geometries is more complex. It presents quite specific challenges regarding interaction strategies between adjacent finite volumes, which impacts in accuracy. The model proposed here can be tested in these dynamic cases before being properly applied to more challenging problems related to linear forming. The results obtained show good agreement with the boundary element solutions.

Furthermore, the Direct Interpolation model can be tested, for the first time, in conjunction with the domain superposition technique to solve dynamics problems in piecewise heterogeneous media. These initial results were encouraging and motivate their application to other, even more challenging cases, which include sectors with gradual changes in heterogeneity, such as those tested, in which the heterogeneity is uniquely sectoral. The numerically successful solution of such heterogeneous problems, in principle unsuitable for a technique based on the boundary discretization, opens up a new branch of research and new applications.

References

[1] K. F. Graff. Wave motion in elastic solids. Dover Publications, 1994.
[2] J. Stewart. *Cálculo*. Pioneira Thomson Learning, 7ed., Vol. 2, 2009.
[3] F. B. Hildebrand. *Methods of Applied Mathematics*. Dover Pub., New York, 1965.
[4] P. C. Oliveira. Esquema FLUX-SPLINE aplicado em cavidades abertas com convecção natural. *Tese de Doutorado em Engenharia Mecânica*, Unicamp, 1997
[5] S. V. Patankar. *Numerical Heat Transfer and Fluid Flow*. Hemisphere Publishing Corporation, 1980.
[6] K. J. Bathe. *Finite Element Procedures in Engineering.* Analysis. Prentice-Hall, 1982.
[7] V. Danu and C. Tenaud. High Order One Step Monotonicity-Preserving Schemes for Unsteady Compressible Flow Calculations, Journal of Computational Physics, vol. 193, pp. 563-594, 2004.
[8] C. A. Brebbia and S. Walker. *Boundary Element Techniques in Engineering*. Elsevier, 1980.
[9] L. O. Lara, C. F. Loeffler, J. P. Barbosa and W. J. Mansur. The technique of domain superposition to solve piecewise homogeneous elastic problems. Engineering Analysis with Boundary Elements, 94, pp. 1-9, 2018
[10] J. P. Barbosa and C. F. Loeffler. Application of Boundary Element Method superposition technique for solving natural frequencies in piecewise homogeneous domains. Computers & Mathematics with applications, 79(4), pp. 1131-1144, 2020.
[11] H. M. Barcelos, C. F. Loeffler and L. O. C. Lara. The direct interpolation boundary element method and the domain superposition technique applied to piecewise Helmholtz's problems with internal heterogeneity. Engineering Analysis with Boundary Elements, 131, pp. 41-50, 2021.
[12] D. Nardini and C. A. Brebbia. *Transient Dynamic Analysis by the Boundary Element Method*. In: C.A. Brebbia Ed, Proceedings of BEM, pp. 719-730, 1983.
[13] M. D. Buhmann. *Radial Basis Functions*. [S.l.]: Cambridge, 2004.
[14] C. F. Loeffler, A. L. Cruz and A. Bulcão. Direct use of Radial Basis Interpolation Functions for Modelling Source Terms with the Boundary Element Method, Engineering Analysis with Boundary Elements. Vol. 50, pp. 97-108, 2015.
[15] T. G. Balista, C. F. Loeffler, L. O. C. Lara, W. J. Mansur. Comparisons Between Direct Interpolation and Reciprocity Techniques of the Boundary Element Method for Solving Two-Dimensional Helmholtz Problems. Engineering Computations. Vol. 40(9/10), pp. 2841-2861, 2023.
[16] G. A. R. SANTOS, C. F. LOEFFLER, A. BULCÃO and LARA, L. O. C. The Direct Interpolation Boundary Element Method for Solving Acoustic Wave Problems in the Time Domain. Computational & Applied Mathematics. Vol 44, pp. 1-30, 2024.