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Use of the equivalent system concept to calculate the natural frequency of buildings considering the effect of axial loading on columns with adjustable restraint

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Abstract. Column stiffness is a fundamental parameter for the dynamic analysis of buildings. One of the most important applications of structural dynamics theory is the use of the equivalent system concept, which allows transforming systems with infinite degrees of freedom into one containing only a single degree of freedom. With this approach, multi-story buildings can be modeled as single-degree-of-freedom systems, allowing the stiffness and mass of the columns to be included in the analysis. In the approach developed in this work, a standard column is assumed to compose the floors of a building, with its properties being calculated based on the generalized coordinate assumed for its boundary conditions. In this context, the column's stiffness is calculated to take into account its geometry, the influence of concentrated and distributed axial loading, and a certain degree of rotational restriction at the base, considered by the application of a rotational spring located at this position. The simulation assumes an arbitrary number of columns per floor and a certain number of floors in the building. A three-dimensional system is therefore represented by an equivalent system obtained by combining springs and masses, first in parallel, representing the grouping of columns on the floors, and then in series, representing the association of the floors within the building. With this strategy, it was possible to determine the equivalent stiffness of each floor as the sum of the column stiffnesses, and that of the floors as the inverse of the sum of the inverses of the floor stiffnesses at the different levels. The equivalent mass is obtained by eigenvalues in relation to the series system. Thus, the natural frequency of the original model could be calculated using the concept of an equivalent system calculated under the influence of the loading, making this a non-linear approach.

Keywords: structural dynamics; association of springs and masses; equivalent stiffness; equivalent mass; natural frequency.

1 Introduction

Buildings are three-dimensional systems with multiple degrees of freedom. As the number of columns per floor and the number of floors in a building increase, the degrees of freedom increase proportionally, making the analysis extremely complex and costly from a mathematical point of view. Therefore, examination techniques must allow the analysis to be performed regardless of the system's degree of complexity, while remaining as practical as possible. One way to achieve this is through the use of analytical-mathematical methods, especially low-order ones, which appear as a counterpoint to the approximate solutions offered by the finite element method (FEM) [1]. It is important to note that analytical methods are classical mathematical procedures capable of providing exact solutions to the problem and that can, but not necessarily, be implemented and solved numerically. This means that the natural frequencies of buildings can be estimated by assuming an equivalent system, formulated analytically, in which the concepts of equivalent stiffness and mass prevail in determining their natural

vibration frequencies [2]-[3]. Particularly, Zalka [3] used a simple analytical method, based on concepts similarly to those developed in this work, for analyzing the frequency of braced tall buildings. In this proposal, the lateral vibration of a building is obtained by an equivalent system that takes into account the "local" bending of columns and the "global" bending of the building, which is influenced by the axial deformations of the columns. Based on the stiffnesses associated with these deformations, a closed-form equation has been elaborated for calculating the lateral frequencies of braced buildings, whose results presented a difference of 7% in comparison with FEM.

In this work, column stiffness is the fundamental parameter of the mathematical approach. Thus, a multi-story building, a system with multiple degrees of freedom [4], can be modeled as one containing only a single degree of freedom. A sophistication found in the present proposal is the **consideration** of **column mass** and the **influence of axial force**. This is a **nonlinear consideration**, as the structural response is calculated from a certain condition that is **no longer** the **initial unloaded** one. It is important to highlight that one of the direct applications of the equivalent system concept is the evaluation of the response of buildings to earthquakes [5] .

In terms of equivalent systems, Urruzola et al. [6] proposed a method that reduces the number of structural elements required for accurate estimates of the natural frequency of frame structures. This efficiency was achieved by correcting the expressions for kinetic energy and elastic deformation of certain components. In their research, Jiang et al. [7] developed a method for evaluating the natural frequencies of timber towers supported by a monopile foundation. The procedure adopted took into account the mass ratio, defined as the ratio of the mass concentrated at the top to the total mass of the tower, and the principles of equivalent stiffness. Nitti et al. [8] presented a low-order analytical formulation for determining the natural frequencies and modes of vibration of tall buildings. The proposal proved particularly useful, especially in the early design stages, offering engineers an effective tool for estimating the dynamic response of these buildings.

Traditionally, multi-story structures have been modeled as systems with multiple degrees of freedom, whose analysis requires complex numerical approaches. For this reason, models that simplify the calculation process are always desirable. In this sense, applying the concept of an equivalent system emerges as a potentially suitable strategy for reducing the effort required without losing the accuracy of the results. **Simplified models** for estimating the natural frequency are specially desired for early-stage design of tall buildings, when FEM is too costly. Another practical motivation is related to earthquake engineering, where equivalent models allow quick screening of structural safety and elaborating vibration control measures, such as by using a digital control system [9] and an active control algorithm associated with a 3 DOF shear building [10]. In this sense, eequivalent system concept could be: (a) coupled with machine learning or reduced-order modeling for rapid estimation of building dynamics; (b) incorporated into data-driven seismic response prediction frameworks, and (c) used in advanced control strategies (e.g., structural vibration suppression, active damping). All these control strategies align with *Emerging Frontiers in Nonlinear Dynamics, Machine Learning, and Control Systems*, including the use of the rotation spring placed at the column base as a regulator for the structural frequency as demonstrated by Wahrhaftig et al. [11].

It is important to emphasize that the equivalent representation of a structure consists of replacing the original, complex model with multiple degrees of freedom with an idealized, simpler one, containing a single degree of freedom, in which masses and stiffnesses are grouped to produce the same effects as the original system. In the case of buildings, this idealization can be achieved by assuming that the floors concentrate mass, while the columns behave as elastic elements similar to springs, which may or may not contribute mass to the system. When computed, the **column masses** add a particular sophistication to the mathematics of the problem.

On the other hand, column-foundation and beam-column connections can be efficiently represented by a rotational spring added to the corresponding boundary condition, which reflects the structural element's reluctance to lateral movement. Thus, the frame structure of a building can be well represented by a combination of springs and masses—first in parallel, in the grouping of columns on the floors, and then in series, in the overlapping floors of the building. In this approach, column **stiffness depends** on its **geometric** and **material properties**, the **stress state induced** by **axial loading**, and **its boundary conditions**, which include the rotational spring added to the base to represent the frame connections.

In this context, this article proposes a methodology for calculating the natural frequency of multi-story buildings, using the concept of equivalent system. This approach considers the total stiffness of the columns, including the influence of rotational springs located at the base, as well as the equivalent masses of the columns and of a certain number of stories. By the present approach, the effect of the axial force is taken into account for establishing the stiffness of the columns as well as its mass, what repercusses on the structural frequency.

2 Problem definition

Consider a building with c columns and p stories, where c and p are natural numbers greater than or equal to 1. Admit the columns, which are resistant to lateral movement, as springs with a given stiffness to be determined and endowed with mass. In the model, the stories are composed of the association of springs in parallel, and the building is composed of the association of stories in series. In condition of parallel springs, the equivalent story stiffness of the story p, k_p is defined as the sum of the stiffnesses of the columns c, as defined in eq. (1).

$$k_{pp} = k_1 + \cdots + k_c \tag{1}$$

In condition of springs in series, the equivalent building stiffness, $k_{eq,}$ is obtained by the inverse of the sum of the inverses of the equivalent stories stiffnesses for p number of floors, as defined in eq. (2).

$$\frac{1}{k_{eq}} = \frac{1}{k_{p1}} + \cdots + \frac{1}{k_{pp}} \tag{2}$$

This makes to be possible to represent a system with multiple degrees of freedom (MDOF) by one containing only a single degree of freedom (SDOF), as illustrated in Fig. 1.

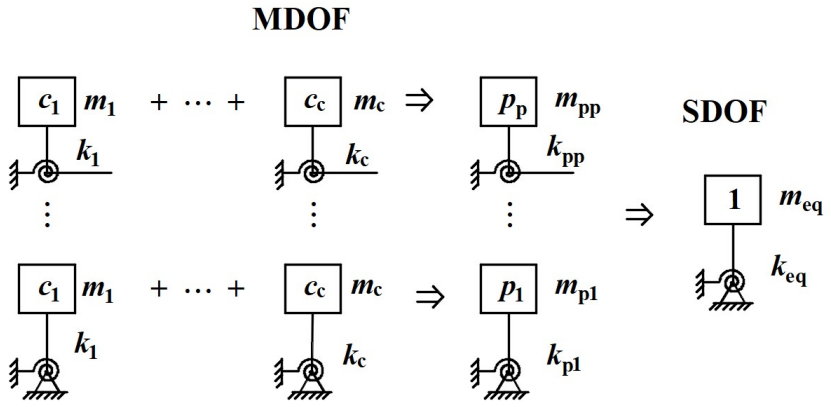


Figure 1. Converting an MDOF system into an equivalent SDOF system. p indicates floor, c columns, m is mass, k is stiffness and eq means equivalent

If the stiffnesses of the columns are equal up to the c-th column, that is, $k_1 = k_2 \cdots = k_c = k$, the stiffness obtained by associating the springs in parallel and in series up to the p-th floor are reduced, respectively, to

$$k_{pp} = kc \quad \text{and} \quad \frac{1}{k_{eq}} = p\frac{1}{k_{pp}} \tag{3}$$

In floors, the column masses are associated in parallel in a similar manner to stiffness. Therefore:

$$m_{pp} = m_1 + \cdots + m_c \tag{4}$$

where m_{pp} is the equivalent mass of the pavement p. If all columns have the same mass, given by m, it can be said that $m_{pp} = mc$ with c being the number of columns on the floor. In the building, the association of masses occurs in series. The study of the equivalent mass by eigenvalues [12] leads to the mass of the building with reference to the p-th floor as:

$$m_{eq} = m_{pp}\left[\left(\frac{1}{1}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{p}\right)^2\right] \tag{5}$$

Thus, the natural frequency of the SDOF system can then be calculated by the relationship between the equivalent stiffness and mass, in the known form, by:

$$\omega_{eq} = \sqrt{\frac{k_{eq}}{m_{eq}}} \text{ (in rad/s)} \quad \text{and} \quad f_{eq} = \frac{\omega_{eq}}{2\pi} \text{ (in Hz)} \tag{6}$$

It is important to be clear that the results obtained from the previous assembly refer to a **non-linear** system because the **stiffness** of the columns is obtained under the **influence** of **applied loads**, both top and distributed along the column, as explained in item 3. By using this way of calculation, the total stiffness of the column in floors is calculated by considering its **two components**: **flexural** stiffness, which depends on the material and cross-section inertia, and **geometric** stiffness, which depends on the acting **normal force**, considered in both **magnitude** and **nature**. Through this approach, the natural frequency becomes **significantly influenced** by the **loading**, which is associated with the geometric stiffness portion of the total generalized stiffness. This means that the stiffness of the column, which is associated with lateral movement, represents its reluctance to change geometry and is considered exclusively geometric. The axial compression force reduces stiffness and vibration frequencies.

3 Column stiffness with adjustable restraint under effect of axial loadings

Consider the column illustrated in Fig. 2, of length l, with a force P applied to the upper end, and having cross-sectional area and inertia given by A and I, respectively. The gravity acceleration is g. The column is made of material with specific weight γ and modulus of elasticity E. A rotational spring with coefficient k is added to its base to represent the imposition of a certain degree of restriction on the rotational motion in that position. When k is equal to zero, the connection is characterized as a hinge, and the column is called hinged-hinged (H-H). If k is infinite, the support will represent a clamp, and the column will be designated as clamped-hinged (C-H). Intermediate values between these two extreme (and idealized) conditions impose a certain level of reluctance of support to the rotational motion. y is an independent geometric variable of the problem.

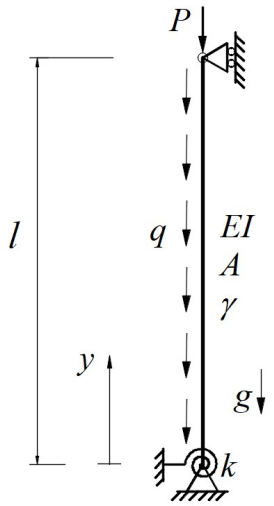
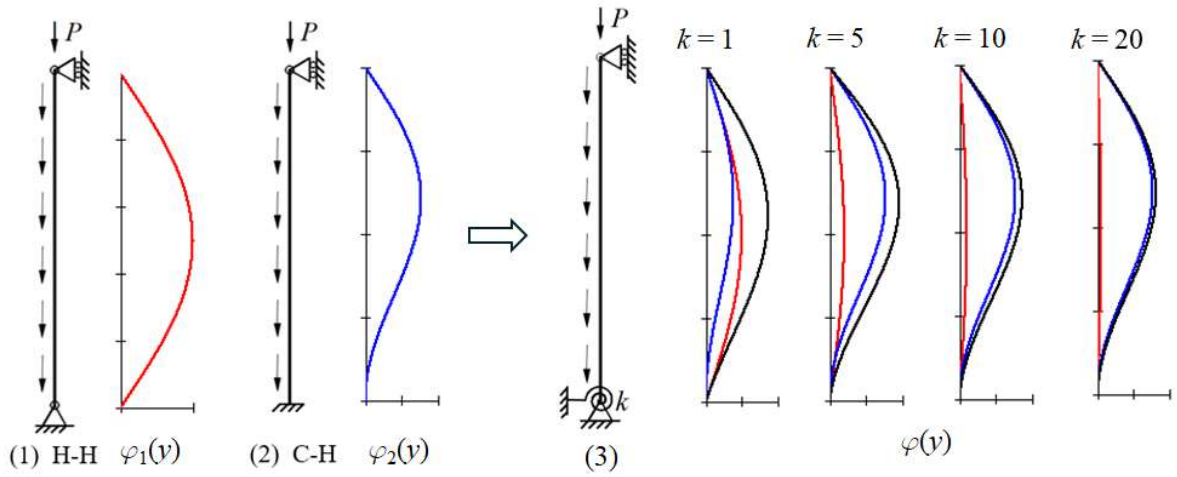


Figure 2. Mathematical model of a column

The vibration forms assumed for column (H-H) and (C-H) can be seen in Fig. 3, condition (1) and (2), respectively, and are represented by the equations indicated below. The intermediate condition (3) between the two previous ones is obtained by superposing the functions corresponding to each of them, being regulated by the rotational stiffness of the spring k, which governs the transition between the two boundary conditions, Fig. 3(3).

In this way, it is possible to calculate the properties of the column of interest of the structural dynamics. According to Wahrhaftig [13]-[14], among these properties are the generalized flexural stiffness, given by eq. (7), and the generalized geometric stiffness, calculated by eq. (8), with the normal force $N(y)$ being known by eq. (9), considering the axially distributed force per unit length q as in eq. (10), where γ is the specific weight of the material, ρ *is* its specific mass and g is the acceleration of gravity.



$$\varphi_1(y) = \sin\left(\frac{\pi y}{l}\right) \qquad \varphi_2(y) = \sin\left(\pi\frac{l-y}{l}\right)\cos\left(\pi\frac{l-y}{2l}\right) \qquad \varphi(y) = \frac{1}{k}\varphi_1(y) + \varphi_2(y)$$

Figure 3. Shape functions for the assumed boundary conditions and transition in between them

$$K_f = \int_0^l EI\left(\frac{d^2\varphi(y)}{dy^2}\right)^2 dy \tag{7}$$

$$K_g = \int_0^l N(y)\left(\frac{d\varphi(y)}{dy}\right)^2 dy \tag{8}$$

$$N(y) = P + \int_y^l q dy \tag{9}$$

$$q = A\gamma \quad \text{with} \quad \gamma = \rho g \tag{10}$$

The stiffness of the rotational spring is obtained by:

$$K_{ri} = k\left(\frac{d\varphi(y_i)}{dy}\right)^2 \tag{11}$$

where *i* indicates the location of the spring along the column. In this case, it will always be zero. *k* is the rotational stiffness coefficient of the spring. The total generalized stiffness of the column is therefore:

$$k = K_f - K_g + K_{r0} \tag{12}$$

and the total generalized mass equal to:

$$m = \left[\int_0^l \frac{A}{\gamma}\left(d\varphi(y)\right)^2 dy\right] + \frac{P}{g}\left(\varphi(l)\right)^2 \tag{13}$$

It is worth noting that the function φ indicated in the previous equations is the one shown in Fig. 3(3). It should be added that for the first floor, the rotational spring represents the connection of the building to its foundation. From then on, for the remaining floors, up to the last, the spring symbolizes the frame nodes, which take into account the restriction imposed by the beam-column connection.

4 Results and discussion

Adopting the prismatic column of Fig. 2 with unitary parameters, such as the cross-sectional inertia, the modulus of elasticity and the specific weight of the material, as well as the rotational stiffness coefficient of the rotational spring, the equivalent stiffness and mass present the behaviors described in Figs. 4(a) and 4(b), respectively, for different numbers of floors *p* and columns *c*. The building frequency, obtained through the equivalent system (SDOF), according to eq. (4), is presented in Fig. 4(c). Once the number of columns and floors is known, the frequency passes to be regulated by the stiffness coefficient of the rotational spring, *k*, making the **frequency** to exhibit strictly **non-linear** behavior, as shown in Fig. 4(d).

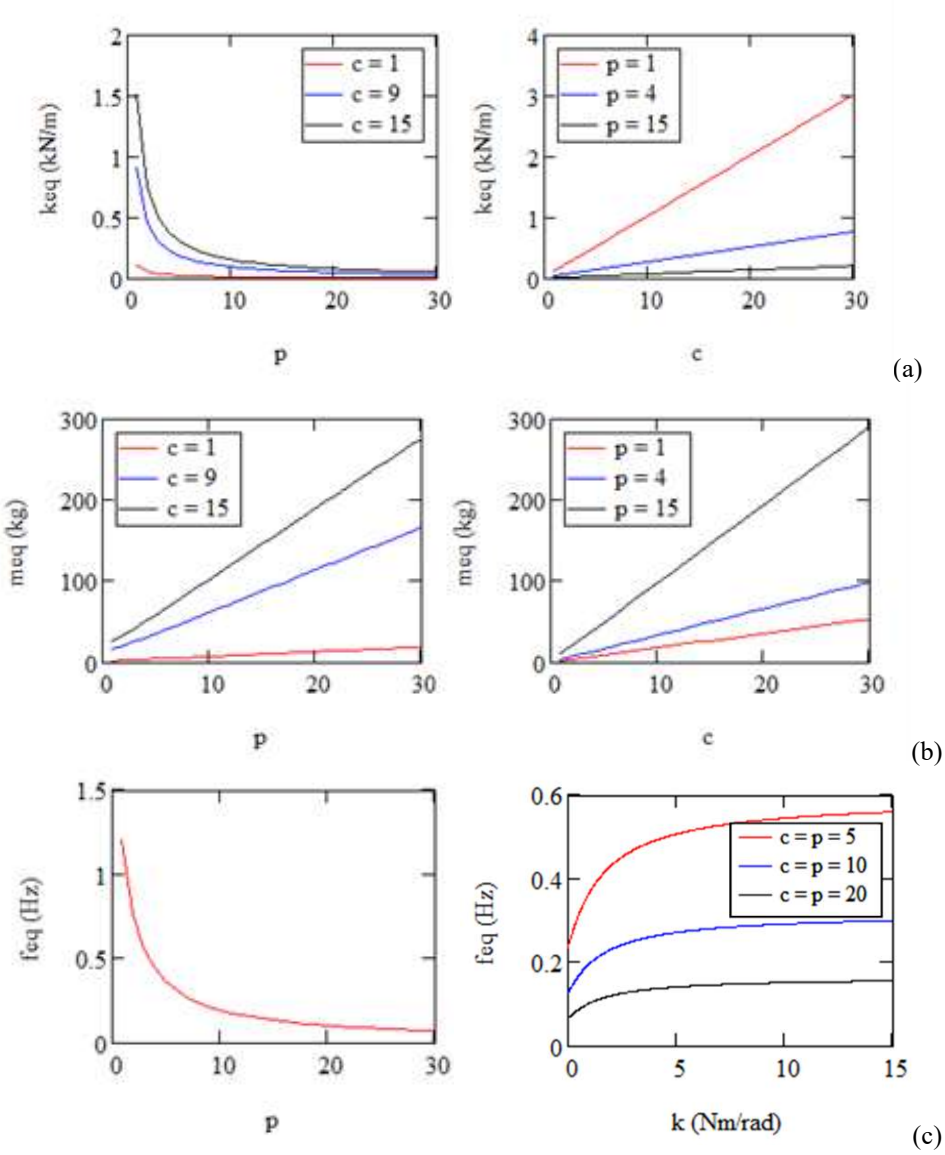


Figure 6. Equivalent system: (a) stiffness, (b) mass, (c) frequency: *p* indicates floor, *c* columns, *m* is mass, *k* is stiffness and *eq* means equivalent

It is possible to verify the existence of **nonlinear behavior** in both the stiffness and the natural frequency of the building with the addition of floors. Stiffness, however, increases linearly with the number of columns in the floor. A similar result is also observed for the equivalent mass. The frequency of the SDOF system is not affected by the number of columns. This occurs because both stiffness and mass are multiplied by the same numerical factor, that is, the number of columns multiplies both the numerator and denominator in the natural frequency calculation, simultaneously affecting the stiffness and equivalent mass of the floor. For practical engineering the

exponential decrease in frequency can take the building to be susceptible quicker to the effect of actions of dynamic nature. On the other hand, this aspect also reveals getting close to collapse at the time its value approaching zero.

5 Conclusions

In this work, a numerical simulation was performed to determine the natural frequency of a building's frame structure using the equivalent system concept. This enabled the transformation of the original model, an MDOF system, into a single-degree-of-freedom model, an SDOF system, enabling primarily qualitative evaluations. It was possible to observe that:

- The natural frequency is significantly influenced by the number of floors in the building, decreasing exponentially as the number of floors increases.
- The equivalent mass of the building is directly proportional to the number of columns and floors, increasing linearly with them.
- The equivalent stiffness of a building increases linearly with the number of columns on a single floor. However, as the number of floors increases, the stiffness decreases in a manner similar to that observed with the equivalent frequency.
- After knowing the number of columns and stories the equivalent frequency passes to be ruled by the coefficient of the rotational spring, imposing a new non-linear behavior.
- The results obtained for the structural frequency of the SDOF can be used for planning an adequate strategy of vibration control, which can be applied to reduce motion whether it came from the wind or earthquakes.
- A strategy for vibration control based on the approach developed in this work is the use of the rotation spring at the column base as a regulator for the structural frequency.
- Vibration modes further than the first can be considered in analysis by altering the shape function.

For future work, the application of the equivalent system concept to evaluate the response of the structural system to dynamic actions such as wind and earthquake is expected. Another relevant aspect worthing highlight is that proprieties of the material such concrete creep also can be considered in an easy way by employing the present analytical strategy. Analytical solutions can provide a quick and useful means of analysis.

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