XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING

On the effect of fractional damping on the nonlinear vibrations of structures

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Abstract. In this work the effect of the fractional-order damping on a continuous system described by the forced Duffing oscillator is studied. For this, numerical methods are implemented and a parametrical study is performed to evaluate the influence of fractional order derivatives on the nonlinear dynamic response. Numerical simulations are implemented in order to study the nonlinear dynamic behavior of bifurcations, chaos, and chaotic attractors and the stability of fixed points is analyzed. It is possible to observe that by increasing the value of the fractional-order parameter, the resulting discrete response is stabilized.

Keywords: fractional damping, duffing oscillator, nonlinear dynamics, chaos, bifurcation analysis.

1 Introduction

The concept of fractional damping has become consolidated as an effective approach for modeling viscoelastic materials and complex mechanical systems, especially those exhibiting nonlinear behavior. This approach is based on the use of fractional derivatives to represent dissipative forces, allowing the incorporation of memory effects and deformation history dependence which are fundamental aspects of the dynamic response of real structures. According to Petráš [1], fractional damping models describe damping forces using fractional derivatives. These models can represent viscoelastic materials more accurately than integer-order models. This accuracy arises from their ability to consider that energy dissipation depends not only on the current state but also on past states of the motion, making them powerful tools for describing complex dynamic systems where traditional integer-order models fail to capture essential dynamics. Indeed, the memory effect, described as a fundamental feature of fractional-order systems, is particularly important in modeling damping and energy dissipation, and is essential for accurately representing hysteresis phenomena and internal losses in frequency-dependent materials.

In this context, Amabili [2] experimentally demonstrates that fractional damping provides a more realistic representation of energy dissipation in viscoelastic materials subjected to large-amplitude vibrations, surpassing classical models by enabling a continuous transition between purely elastic behavior ($\alpha = 0$) and purely viscous behavior ($\alpha = 1$). Furthermore, according to Amabili [3], the damping force expressed as $cD^{\alpha}x$, with $0 < \alpha < 1$, gives the system the flexibility needed to accurately describe complex phenomena observed in composite and biological structures, enabling the modeling of systems with time- and frequency-dependent dissipations. Finally, studies conducted by Amabili *et al*. [4] reinforce this perspective by applying the fractional model to viscoelastic plates subjected to nonlinear vibrations, demonstrating that the fractional approach improves the prediction of dynamic frequency response and increases the fidelity of models when compared to experimental data.

This work aims to numerically analyze the influence of fractional damping on the nonlinear dynamic response of a clamped-free beam subjected to a base acceleration. To this end, numerical methods capable of simulating structural behavior with different fractional orders α and different initial base displacements y_0 will be implemented. A parametric study will be conducted to evaluate the effects of these variations on the system dynamics, with emphasis on the identification of bifurcations and transitions between unstable and stable regimes.

2 Mathematical Formulation

Consider a clamped-free beam of height H, density ρ, cross-section area A, Young modulus E, and moment of inertia I and subjected to a harmonic base displacement $Y(t) = y_0 \cos(\Omega t)$, as seen in Fig. 1. Variable z corresponds to the spatial coordinate along the length of the beam and $v(z, t)$ the transverse displacement.

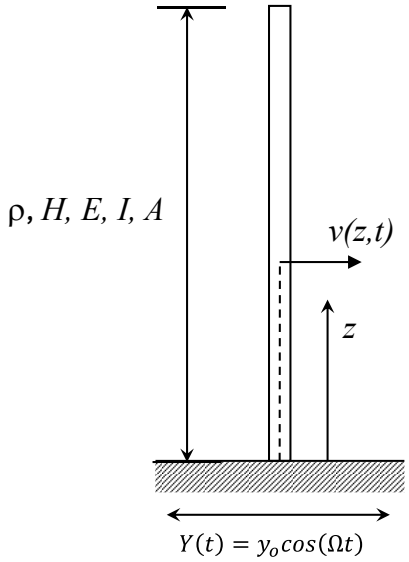


Figure 1. Clamped-free beam

Considering the Euler-Bernoulli nonlinear beam theory, the strain energy U is given by:

$$U = \int_0^H \frac{1}{2} EI \left(v_{,zz}{}^2 + v^2{}_{,zz}\, v^2{}_{,z} + \frac{1}{4} v^2{}_{,zz}\, v^4{}_{,z} \right) dz \tag{1}$$

Considering the harmonic base displacement, the kinetic energy is:

$$T = \int_0^H \frac{1}{2} \rho A \left(v_{,t} + y_{,t} \right)^2 dz \tag{2}$$

The work of the viscous nonconservative forces is:

$$W_{nc} = \int_0^H \frac{1}{2} c v_{,t}{}^2 dz \tag{3}$$

Where c is the damping coefficient.

The Langragean functional of the system can be written as:

$$\mathcal{L} = T - U + W \tag{4}$$

The field displacement satisfying the geometric boundary conditions is given by the eq. (5):

$$v(z,t) = \sum_{i=1}^{m} a_j(t) \left[1 - cos\left(\frac{(2i-1)\pi z}{2H} \right) \right] \tag{5}$$

The coefficients $a_j(t)$ represent displacement amplifications that vary over time, indicating the dynamic response of the system and parameters m represent the number of modes considered in the analysis. By substituting

field displacements of eq. (5) into eq. (1), eq. (2) and eq. (3) applying the Rayleigh-Ritz method together with the Hamilton principle, a set of nonlinear dynamic equilibrium equations are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{a}_i}\right) - \frac{\partial \mathcal{L}}{\partial a_i} = -\frac{\partial W_{nc}}{\partial \dot{a}_i} \quad i = 1..m \tag{6}$$

Considering m = 1, the resulting non-linear equation with viscous damping is given by:

$$\begin{aligned} 0.2267\rho AL\frac{d^2a_1(t)}{dt^2} + 2\xi\rho AL\omega_o\frac{da_1(t)}{dt} + 3.0440\frac{EI}{L^3}a_1(t) + 3.7554\frac{EI}{L^5}a_1(t)^3 \\ + 1.7374\frac{EI}{L^7}a_1(t)^5 = -0.3634\rho AL\frac{d^2Y}{dt^2} \end{aligned} \tag{7}$$

where ξ is the damping coefficient and ω_o is the natural frequency given by:

$$\omega_o = 3.6639\sqrt{\frac{EI}{\rho AL^4}} \tag{8}$$

When fractional damping is going to be considered, the term related to viscous damping in eq. (7) should be modified as:

$$\begin{aligned} 0.2267\rho AL\frac{d^2a_1}{dt^2} + 2\xi\rho AL{\omega_o}^{2-\alpha}\frac{d^\alpha a_1}{dt^\alpha} + 3.0440\frac{EI}{L^3}a_1 + 3.7554\frac{EI}{L^5}a_1(t)^3 \\ + 1.7374\frac{EI}{L^7}a_1(t)^5 = -0.3634\rho AL\frac{d^2Y}{dt^2} \end{aligned} \tag{9}$$

where $\frac{d^\alpha a_1}{dt^\alpha}$ is the fractional derivative of order α.

Eq. (7) can be easily solved numerically by applying the fourth-order Runge-Kutta method and can be written as a first-order system of differential equations in the form:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t) \\ \frac{dx_2(t)}{dt} &= \frac{1}{0.2267\rho AL}\left(-2\xi\rho AL\omega_o x_2(t) - 3.0440\frac{EI}{L^3}x_1(t) - 3.7554\frac{EI}{L^5}x_1(t)^3\right. \\ &\quad \left. - 1.7374\frac{EI}{L^7}x_1(t)^5 - 0.3634\rho AL\frac{d^2Y}{dt^2}\right) \end{aligned} \tag{10}$$

On the other hand, eq. (9) can be written as a first-order system of differential equations in the form:

$$\begin{aligned} {}_0D_t^{q_1}x_1(t) &= x_2(t) \\ {}_0D_t^{q_2}x_2(t) &= \frac{1}{0.2267\rho AL}\left(-2\xi\rho AL{\omega_o}^{2-\alpha}x_2(t) - 3.0440\frac{EI}{L^3}x_1(t) - 3.7554\frac{EI}{L^5}x_1(t)^3\right. \\ &\quad \left. - 1.7374\frac{EI}{L^7}x_1(t)^5 - 0.3634\rho AL\frac{d^2Y}{dt^2}\right) \end{aligned} \tag{11}$$

where q_1 and q_2 are two fractional orders.

The numerical calculation of the fractional order derivatives can be given by Eq. (12), which is derived from the Grünwald-Letnikov fractional order derivative. The relation for the explicit numerical approximation of *q-th* derivative at the points *kh*, (*k*=1, 2, …) has the following form [1]:

$${}_{(k-L_m/h)}D_{t_k}^q f(t) \approx h^{-q}\sum_{j=0}^{q}(-1)^j\binom{q}{j}f(t_{k-j}) \tag{12}$$

where L_m is the "memory length", $tk = kh$, h is the time step and $(-1)^j\binom{q}{j}$ are binomial coefficients. For their $c_j^{(q)}(j = 0, 1, \dots)$ calculations the following expression can be used:

$$c_0^{(q)} = 1, \quad c_j^{(q)} = \left(1 - \frac{1+q}{j}\right)c_{j-1}^{(q)} \tag{13}$$

The numerical solution of a general fractional differential equation:

$${}_aD_t^q y(t) = f(y(t), t) \tag{14}$$

is given by:

$$y(t_k) = f(y(t_k), t_k)h^q - \sum_{j=v}^{k} c_j^{(q)} y(t_{k-j}) \tag{15}$$

For the memory term, the "short memory" principle can be used. The lower index of the sums in eq (15) will be $v = 1$ for $k < (L_m/h)$ and $v = k-(L_m/h)$ for $k > (L_m/h)$.

Finally, the numerical solution of the fractional-order eq. (11), has the following form:

$$\begin{aligned} x_1(t_k) &= x_2(t_{k-1})h^{q_1} - \sum_{j=v}^{k} c_j^{q_1} x_1(t_{k-j}) \\ x_2(t_k) &= \left\{ \frac{1}{0.2267\rho AL} \left(-2\xi\rho AL{\omega_o}^{2-\alpha} x_2(t_{k-1}) - 3.0440\frac{EI}{L^3}x_1(t_k) - 3.7554\frac{EI}{L^5}x_1(t_k)^3 \right.\right. \\ &\quad \left.\left. - 1.7374\frac{EI}{L^7}x_1(t_k)^5 - 0.3634\rho AL\frac{d^2Y}{dt^2} \right) \right\} h^{q_2} - \sum_{j=v}^{k} c_j^{q_2} x_2(t_{k-j}) \end{aligned} \tag{16}$$

where T_{sim} is the simulation time, $k = 1, 2, 3, .., N$, for $N = (T_{sim}/h)$, h = time step and $x_1(0)$ and $x_2(0)$ the initial conditions.

3 Numerical Results

Consider a clamped-free beam with the following physical and geometrical properties: $\rho = 7850\ kg/m^3$, Young modulus $E = 153.4011045\ GPa$, cross section $A = 0.23411\text{e-}4\ m^2$ and moment of inertia $I = 1.3118\text{e-}12\ m^4$. The beam is subjected to increasing values of base displacement. To study the influence of damping on the nonlinear response of the beam, the resonance curves will be obtained for increasing values of base displacement (y_0) considering the viscous damped system given by eq. (10) and the fractional order system given by eq. (11).

First, Fig. 2 displays the resonance curves of the viscous damped system obtained for $\xi = 0.001$ considering increasing values of base displacement. As can be observed, the beam displays typical linear behavior for small values of base displacement and hardening behavior with large amplitude 1T vibrations for large base displacements.

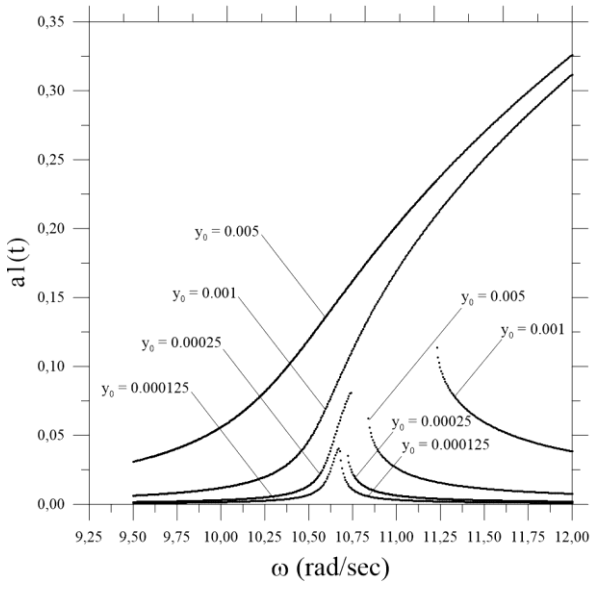
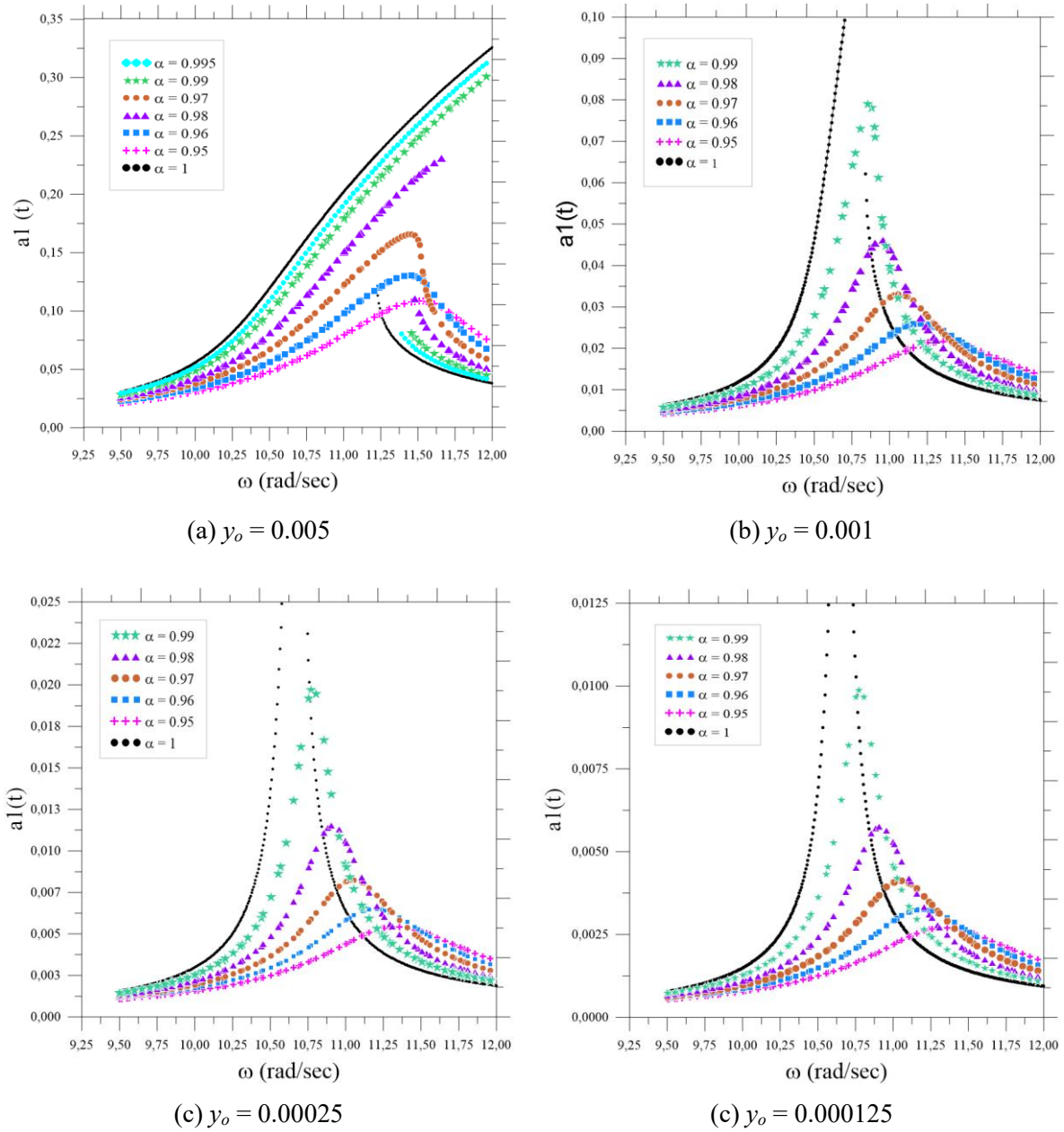


Figure 2. Resonance curve of viscous damped system

Now, the influence of the fractional damping will be considered, for this, the fractional derivative will be varied from $\alpha = 0.99$, $\alpha = 0.98$, $\alpha = 0.97$, $\alpha = 0.96$ and $\alpha = 0.95$ and considering four values of base amplitude oscillation $y_0 = 0.005$, $y_0 = 0.001$, $y_0 = 0.00025$, and $y_0 = 0.000125$.

These simulations were conducted using a fractional-order numerical scheme based on the Grünwald–Letnikov approximation, implemented in Maple and MATLAB, as described by Petráš [1], and adapted for the nonlinear Duffing dynamics derived from the Hamiltonian formulation presented in this work. The governing equation includes nonlinear stiffness terms (cubic and quintic), and fractional damping is introduced as a memory-dependent term that modulates energy dissipation with the derivative order α.



(a) $y_o = 0.005$ (b) $y_o = 0.001$

(c) $y_o = 0.00025$ (c) $y_o = 0.000125$

Figure 3. Comparison of resonance curves for varying α and y_0

Figure 3 presents the comparison of the resonance curves of the clamped-free beam. Figure 3(a) displays the resonance curves for $y_0 = 0.005$ *m*. As seen for α = 1 (viscous damped system), the system shows strong hardening behavior but, when fractional damping is considered, there is a strong change on the nonlinear dynamic behavior. For α = 0.99 and α = 0.98 there is still hardening behavior with smaller amplitudes than the viscous system but, when α goes to 0.99, 0.96 and 0,95 the hardening behavior is not displayed, and the peaks of the curves are shifted to the right. Also, the fractional system displays larger amplitude vibrations than the viscous system after the resonance point.

Figure 3(b) displays the resonance curves for $y_0 = 0.001\ m$. Again, the viscous damped system displays hardening behavior but, when fractional damping is considered, the hardening behavior disappears, and all vibration amplitudes are reduced. Again, it is also possible to see that the fractional system displays larger amplitude vibrations than the viscous system after the resonance point. Figures 3(c) and 3(d) display the comparison of the resonance curves for $y_0 = 0.00025\ m$ and for $y_0 = 0.000125\ m$. As can be seen, when fractional damping is considered, all curves display smaller vibration amplitudes and are shifted to the right, with larger vibrations than the viscous system after the resonance point.

4 Conclusions

In this work the influence of fractional damping on the nonlinear vibrations of a clamped-free beam subjected to harmonic base displacement was studies. For this, the Euler-Bernoulli nonlinear beam theory was applied, and the resulting equations were solved numerically by applying the fourth-order Kunge-Kutta method for the viscous system and the fractional system was solved by applying the Grünwald-Letnikov fractional order derivative.

The nonlinear resonance curves were obtained for increasing values of base displacement amplitudes and for reducing values of fractional derivative. As can be seen, for the viscous damped system, the beam displays typical hardening behavior for large base displacements but, when fractional damping is considered, there is a reduction on the vibration amplitudes for reducing values of the fractional derivatives which means that, depending on the damping consideration, the system will display smaller vibration amplitudes. Also, it is possible to see that the fractional systems displays larger amplitude vibrations than the viscous system after the resonance point.

The main contribution of this research lies in demonstrating that the use of fractional derivatives not only improves the modeling of energy dissipation but also directly influences the dynamic stability of nonlinear systems, offering new pathways for the design of safer, more durable, and energy-efficient structures. Nonlinear effects become more pronounced for larger initial base displacements, as shown in Fig. 3(a), leading to the emergence of bifurcations, multiple responses, and loss of stability. As the initial displacement decreases, as seen in Fig. 3(b), Fig. 3(c), and Fig. 3(d), the system tends to progressively linearize and restore global stability. Fractional damping acts as an efficient stabilizing mechanism, reducing the response amplitude, smoothing the resonance curve, and expanding the range of stable frequencies. The transition from an unstable regime (with bifurcations and chaos) to a stable regime occurs gradually and is influenced by the combination of the fractional order and the magnitude of the initial displacement. The shift of the resonance peak with increasing amplitude indicates stiffening behavior, characteristic of Duffing-type systems with positive cubic nonlinearity. The elimination of chaotic trajectories at lower amplitudes, as shown in Fig. 3(d), further highlights the role of fractional damping as a dynamic control tool.

Additionally, it is observed that as the fractional order α moves away from 1 (i.e., decreases toward 0), the system's behavior gradually transitions from a viscous-dominated regime to a memory-dominated regime. In general, lower values of α tend to increase system stability by enhancing dissipative effects, especially under low-excitation conditions. Conversely, α values closer to 1 may reduce the influence of memory effects, favoring the onset of unstable and chaotic dynamics in high-excitation regions. These results demonstrate that the interaction between the fractional derivative order and the system's initial condition significantly affects its dynamic response. The analysis of resonance curves allows for a better understanding of the boundaries between stable and unstable regimes, and highlights the role of memory effects introduced by fractional derivatives in modulating nonlinear structural responses under periodic excitations.

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