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Implementations for the GFEM global-local method on the INSANE platform - Mathematical determination of the stiffness parameter

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Abstract. The Generalized Finite Element Method with global-local enrichment (GFEM^{gl}) builds enriched approximation functions using the standard Global-Local Finite Element framework. The method involves three stages: (1) solving the global problem over the entire domain, (2) solving a local problem to capture detailed phenomena, and (3) returning to the global analysis, now informed by local results. A key challenge in this process is defining the stiffness parameter that links global and local models, which is typically chosen empirically based on material and boundary conditions. This work proposes a mathematical formulation to determine the stiffness parameter, eliminating the need for empirical tuning. The method follows the approach of Birner and Schweitzer [1], estimating the parameter by ensuring that the global force and displacement vectors are of comparable magnitude. This formulation was implemented in INSANE, an in-house finite element platform developed at UFMG. To assess the method, a simulation of crack propagation in a two-dimensional linear elastic medium were performed, in Mixed-Mode fracture case. The result were compared with a reference solution, showing that the proposed approach is effective and avoids relying on an empirical equation to calculate the stiffness parameter.

Keywords: Generalized/extended Finite Element Method , Global-local analysis, Linear Elastic Fracture Mechanics, Penalty Method

1 Introduction

The Generalized Finite Element Method with Global-Local Enrichments, GFEM^{gl}, was first proposed by Duarte and Kim [2], utilizing the global-local scheme of standard FEM [3], within the enrichment strategy approach of GFEM [4, 5]. In GFEM^{gl}, the solution of the problem is performed considering two-scale problems. The global problem, represented by a coarse discretization of the structure, is responsible for representing the global behavior of the problem. The local problem, restricted to a small portion of the problem, describes, using a fine discretization, confined phenomena of interest. Both problems are solved separately. The coupling between them is established by global data transferred as boundary conditions to the local problem and the enrichment of the global approximation with the numerical solution of the local problem. The quality of these boundary conditions is a key aspect of the performance of GFEM^{gl}. Initially, only Dirichlet boundary conditions were imposed on the local problem. Aiming to provide more accurate solutions, Kim et al. [6] proposed to use Cauchy/spring boundary conditions. In this approach, the authors introduce an empirical parameter κ to combine the imposition of the global fields of stress and displacements as boundary conditions for the local problem. An automatic scheme to obtain an optimal value for κ is proposed by Birner and Schweitzer [1]. Finally, Marques et al. [7] use a recovered stress field to enhance the data transferred from global to local-scale models. In this approach, the recovered stresses are computed by solving a block-diagonal system of equations presented by Lins et al. [8], based on the method proposed originally by Zienkiewicz and Zhu [9].This work implements and evaluates, in one crack problem, the methodology proposed by Birner and Schweitzer[1] on the INSANE platform [10], available on https://www.insane.dees.ufmg.br. The current implementation of GFEM, based on Alves et al.[11], utilizes the global-local strategy to simulate crack propagation in one-dimensional problem as Fonseca [12] automated.

2 Generalized Finite Element Method with Global-Local Enrichments

Following Kim et al. [6] the strategy is divided in three stages that can be divided in cycles to improve the accuracy of the solution, according to O'Hara et al.[13].

Stage 1: Global initial problem:

The problem domain is created with a coarse finite element mesh. At this stage, is not necessary to represent the local features of the problem, such as discontinuities and singularities.

The approximate solution of the global problem can be obtained by solving eq.(1) in which Ω_{G} corresponds to the global domain,$\boldsymbol{\sigma}$ to Cauchy stress tensor, $\boldsymbol{\epsilon}$ to the Small strain tensor, $\bar{\boldsymbol{t}}$ to the prescribed stress vector applied on the Neumann boundary Ω_G^{σ}, $\tilde{\boldsymbol{u}}_G^0$ on the approximate solution of the global initial problem, that satisfies the Dirichlet boundary conditions on Ω_G^u and υ_G^0 as test function.

$$\int_{\Omega_G} \boldsymbol{\sigma}(\tilde{\boldsymbol{u}}_G^0) : \boldsymbol{\epsilon}(\boldsymbol{\upsilon}_G^0) d\omega = \int_{\partial\Omega_G^{\sigma}} \bar{\boldsymbol{t}} \cdot \boldsymbol{\upsilon}_G^0 d\gamma \tag{1}$$

The GFEM approximate solution of the global initial problem $\tilde{\boldsymbol{u}}_G^0$ is obtained by eq.(2) with N_j representing the partition of unity functions of the standard finite element method for the set of nodal points η in the problem domain, the functions L_{ji} represent the set of local enrichment functions of the approximation, q_j are the number of GFEM enrichment functions corresponding to the node j, u_j is the nodal parameter from FEM and the b_{ji} is the nodal parameter from GFEM.

$$\tilde{\boldsymbol{u}}_G^0 = \sum_{j=1}^{n} \mathrm{N}_j \left\{ \mathbf{u_j} + \sum_{i=2}^{\mathrm{q}_j} \mathrm{L}_{ji} \mathbf{b_{ji}} \right\} \tag{2}$$

Stage 2: Local problem:

At the neighborhood of the local feature of interest, a new problem with a fine discretization is created. The boundary conditions of the problem are defined using the results from Stage 1 and the local domain is defined in a way to describe the singularity, using a refined mesh and enrichment functions. As proposed by Moes et al[14] and Park et al[15], those functions are used to represent discontinuities and singularities in the tension field around a crack. The singularity functions represent the tension discontinuity on crack tip and can be described as asymptotics functions, proposed by Szabo and Babuska[16] and used on Oden and Duarte[17] paper. These functions depend on the crack opening mode, the coordinates of the crack tip and the material geometrical properties. The eq.(3) and eq.(4) represent Mixed-Mode crack opening, being G the transversal elasticity constant, $\kappa = 3 - 4\nu$ for plane strain, $\lambda^{(1)} = \lambda^{(2)} = 0,5$,$\mathrm{Q}^{(1)} = 0,333$,$\mathrm{Q}^{(2)} = -1,0$,r and θ are crack tip polar coordinates.

$$\mathrm{u}_x^{(2)} = \frac{1}{2G} \mathrm{r}^{\lambda(2)} \left[(k - \mathrm{Q}^{(2)}(\lambda^{(2)} + 1)) \sin \lambda^{(2)}\theta - \lambda^{(2)} \sin(\lambda^{(2)} - 2)\theta \right] \tag{3}$$

$$\mathrm{u}_y^{(2)} = \frac{1}{2G} \mathrm{r}^{\lambda(2)} \left[(k - \mathrm{Q}^{(2)}(\lambda^{(2)} + 1)) \cos \lambda^{(2)}\theta - \lambda^{(2)} \cos(\lambda^{(2)} - 2)\theta \right] \tag{4}$$

The approximate solution of the local problem can be obtained by eq.(5), with Ω_{L} corresponding to local domain, $\boldsymbol{\sigma}$ to Cauchy stress tensor, $\boldsymbol{\epsilon}$ to the Small strain tensor, $\bar{\boldsymbol{t}}$ to the prescribed stress vector applied on the Neumann boundary Ω_L^{σ}, $\tilde{\boldsymbol{u}}_L$ on the approximate solution of the local problem, that satisfies the Dirichlet boundary conditions on Ω_L and υ_L as test function.

$$\int_{\Omega_L} \boldsymbol{\sigma}(\tilde{\boldsymbol{u}}_L) : \boldsymbol{\epsilon}(\boldsymbol{\upsilon}_L) d\omega + \kappa \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} \tilde{\boldsymbol{u}}_L \cdot \boldsymbol{\upsilon}_L d\gamma = \int_{\partial\Omega_L \cap \partial\Omega_G} \bar{\boldsymbol{t}} \cdot \boldsymbol{\upsilon}_L d\gamma + \int_{\partial\Omega_L \setminus (\partial\Omega_L \cap \partial\Omega_G)} (\boldsymbol{t}(\tilde{\boldsymbol{u}}_G^0) + \kappa \tilde{\boldsymbol{u}}_G^0) \cdot \boldsymbol{\upsilon}_L d\gamma \tag{5}$$

The approximate solution for $\tilde{\boldsymbol{u}}_L$ is given by eq(6) with $\boldsymbol{b_{ji}}$, $\boldsymbol{c_{ji}}$ and $\boldsymbol{d_{ji}}$ as nodal parameters and $\boldsymbol{u}^{(1)}$ and $\boldsymbol{u}^{(2)}$ diagonal matrices containing (3-4) respectively.

$$\tilde{\boldsymbol{u}}_L = \sum_{j=1}^{n} \mathrm{N}_j \mathbf{u_j} + \sum_{j} \mathrm{N}_j \left\{ \mathbf{u^{(1)}} \mathbf{b_{ji}} + \mathbf{u^{(2)}} \mathbf{c_{ji}} \right\} \tag{6}$$

The process to define the boundary conditions of the local problem, according to Kim et al[6], uses the stiffness parameter κ.

The boundary conditions can be transferred in three different ways:

1. Newmann boundary conditions: in this case, $\kappa = 0$ and therefore just stresses are transferred from the global problem to the local problem.
2. Dirichlet boundary conditions: $\kappa = \eta >> 1.0$, corresponding to the penalty method parameter. In practice, only the displacements are transferred as boundary conditions for the local problem.
3. Cauchy boundary conditions: κ is determined as an intermediate value between conditions 1 and 2 above. Thus, both boundary conditions (stresses and displacements) are transferred to the local problem. Based on numerical experiments, Kim et al[6] propose determining κ according to eq.(7), in which E is the Young's modulus, V_0 is the volume of parametric element, n is the problem spatial dimension (here $n = 2$), J is the Jacobian of the global element that has edges on the boundary of the local problem.

$$\kappa = \frac{E}{\sqrt[n]{\mathrm{V}_0 J}} \tag{7}$$

Using eq.(7) the κ parameter is determined empirically. The methodology proposed by Birner and Schweitzer [1] for the numerical determination of the stiffness parameter, which is the focus of this work, is discussed in Section 3.

Stage 3: Enriched global problem:

The local problem solution $\tilde{\mathrm{u}}_L$ is used to enrich the global problem. In this way, the local features are represented in the global problem by the enrichment of the approximation with the local numerical solution. The global problem will then include, in addition to the shape functions from the initial global problem, another shape function as defined in eq.(8), in which $\tilde{\boldsymbol{u}}_L$ is a diagonal matrix with the numerical solution from (5).

$$\tilde{\boldsymbol{u}}_G^E = \sum_{j=1}^{n} \mathrm{N}_j \left\{ \mathbf{u_j} + \sum_{i=2}^{\mathrm{q}_j} \mathrm{L}_{ji} \mathbf{b_{ji}} + \mathbf{e_j} \tilde{\boldsymbol{u}}_L \right\} \tag{8}$$

The solution of the enriched global problem $\tilde{\mathrm{u}}_G^E$ is given by eq.(9), with Ω_G corresponding to global domain, $\boldsymbol{\sigma}$ to Cauchy stress tensor, $\boldsymbol{\epsilon}$ to the Small strain tensor, $\bar{\boldsymbol{t}}$ to the prescribed stress vector, $\tilde{\boldsymbol{u}}_G^E$ on the approximate solution of the global problem and υ_G^E as test function.

$$\int_{\Omega_G} \boldsymbol{\sigma}(\tilde{\mathbf{u}}_{\boldsymbol{G}}^{\boldsymbol{E}}) : \boldsymbol{\epsilon}(\boldsymbol{\upsilon}_G^E) d\omega = \int_{\partial\Omega_G^\sigma} \bar{\boldsymbol{t}}.\boldsymbol{\upsilon}_G^E d\gamma \tag{9}$$

3 Model problem and κ formulation

3.1 Numerical determination of κ

The determination of the stiffness parameter as proposed by Birner and Schweitzer [1] is associated with Cauchy boundary conditions, in which both force and displacement are prescribed. The goal is to ensure consistency between the forces and displacements obtained in the global problem with those imposed on the boundary of the local problem. This concept is represented in eq. (10).

$$\boldsymbol{t}(\mathbf{u_L}) + \kappa \mathbf{u_L} = \boldsymbol{t}(\mathbf{u_G}) + \kappa \mathbf{u_G} \tag{10}$$

According to Birner and Schweitzer[1], from a stability standpoint, both terms on each side of the equation must be comparable in magnitude. Therefore, the vector of prescribed stresses in the local model, based on global results (t(u_G)) and the displacement vector from the global model need to be properly scaled. The proposed methodology consists of determining two vectors. Equations (11) and (12) show this vectors adapted to the formulation used in this work in which κ from 1 to n corresponds to the local problem degrees of freedom.

$$\mathbf{M_T(u_G)} = (\int_{\Gamma \mathrm{L}/\Gamma \mathrm{G}} \boldsymbol{t}(\mathbf{u_G}) \mathrm{a}_\kappa \mathrm{ds})_{\kappa=1}^{n} \tag{11}$$

$$\mathbf{M_D(u_G)} = (\int_{\Gamma \mathrm{L}/\Gamma \mathrm{G}} \mathbf{u_G} \mathrm{a}_\kappa \mathrm{ds})_{\kappa=1}^{n} \tag{12}$$

The term a_k corresponds to the basis functions of the approximation in the global problem. Once these vectors are determined over the global problem, the stiffness parameter is obtained by balancing the norms of the vectors, as shown in eq.(13).

$$\kappa(\mathbf{u_G}, \boldsymbol{\sigma}(\mathbf{u_G})) = \frac{\| \mathbf{M_t(u_G)} \|_{L^2}}{\| \mathbf{M_D(u_G)} \|_{L^2}} \tag{13}$$

This expression differs from the empirical formulation proposed by Kim et al[6], eq.(7) in which κ is scaled solely based on material data and the size of the global element with an edge on the boundary of the local problem. In the formulation proposed in eq.(13), the value of κ depends on the forces and displacements obtained from the global problem.

The numerical process for determining the stiffness parameter according to the mathematical formulation described above, was implemented in the INSANE platform. This implementation is based on the automated global-local strategy previously developed by Fonseca[12], and employs a recovered stress field to build the boundary conditions according to Marques [7].

3.2 Test - Crack propagation - Mixed-mode

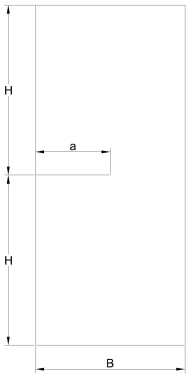
In order to evaluate the strategy discussed on Section 3.1, a problem from the Linear Elastic Fracture Mechanics are considered. A plate on plane strain with an edge crack with size "a" is subject to shear (Mixed-mode). Figure 1(a) shows the geometry of the problems. The material properties are $E = 100 \cdot 10^3$ and $\nu = 0.3$, in consistent units.

Figure 1(b) and Figure 2 shows the meshes adopted in each stage of the GFEM$^{\mathrm{gl}}$ methodology. The initial global mesh consists of 105 four-node quadrilateral elements, with dimensions 1.0667×1.0. The local mesh consists of 135 four-node quadrilateral elements with dimensions 0.333×0.356. The nodes of the enriched global solution are defined automatically and are shown in green in Figure 1(b). This enrichment is also performed automatically, as proposed by Fonseca[12].

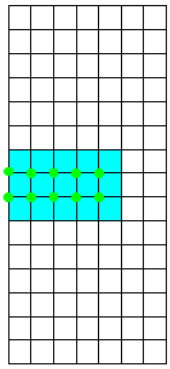
In order to compare the methodology implemented to calculate κ with the stiffness parameter as defined by Kim et al[6], the tests on INSANE platform were conducted changing the value of κ as: 10^1, 10^2, 10^3 ,10^4, 10^5, 10^6, 10^8, 10^{10}. The value of $\kappa = 10^5$ is the result obtained using eq.(7) with the current problem data.

To compare the results, a referenced solution were used for the strain energy with a fine mesh U_{ref}. The result obtained for strain energy on the tests ($\tilde{U}$) were compared using an estimative to the relative error according to eq.(14).

$$\mathrm{e}_U = \log\left(\frac{\mathrm{U}_{ref} - \tilde{U}}{\mathrm{U}_{ref}} \cdot 100\%\right) \tag{14}$$



((a)) Plate - under plane strain - geometry - $H = 8.0, B = 7.0, a = 3.5$ in consistent units



((b)) Global problem with the local subdomain and the nodes (in green) associated with the numerically obtained local solution $\boldsymbol{u_L}$.

Figure 1. Global problem

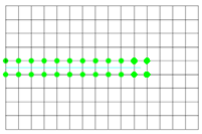


Figure 2. Local enrichment Mixed-Mode - Green - Singularity (eq.(3) to eq.(4))

The results obtained for K_I and K_{II} were also compared with a reference solution and the error were estimated the same way as eq.(14).

The problem on mixed-mode evaluated were proposed by Wilson[18]. As a reference solution, the one obtained by Fonseca[12] using the ANSYS® software was adopted, where the mesh was refined until convergence was achieved, which occurred with 76,184 elements type Q8, resulting in $\mathrm{U}_{ref} = 0.02467211$. Figure 3 illustrates the variation of the error for global-local cycles 1 to 4, for the tested parameter κ values, for the strain energy, Figure 4 lustrates the error for K_I and Figure 5 the error for K_{II}.

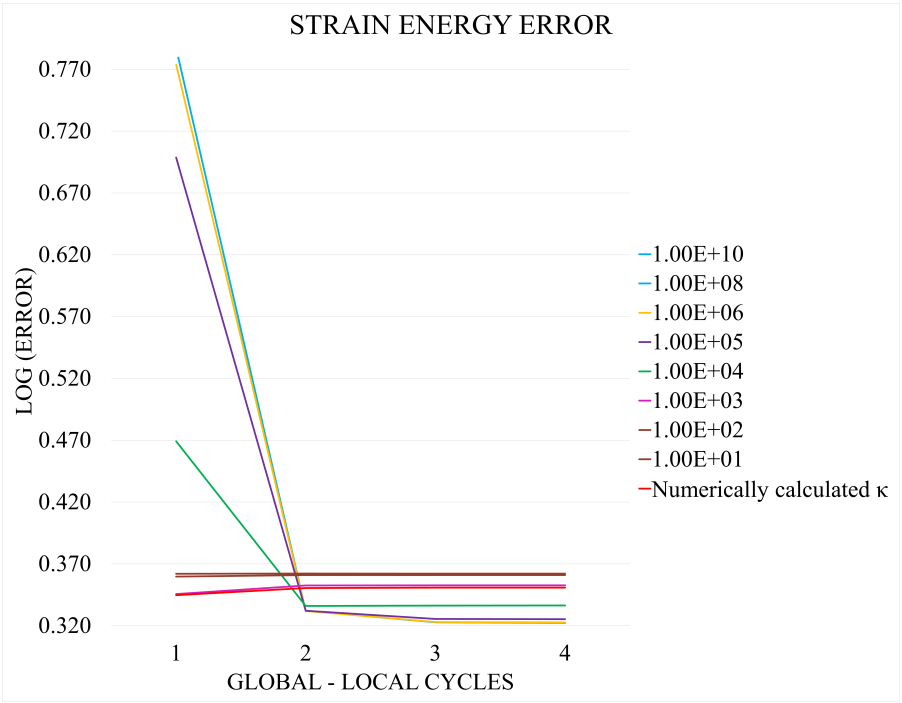


Figure 3. Mixed-Mode Crack – Strain Energy error as a function of κ - Cycles global-local 1 to 4

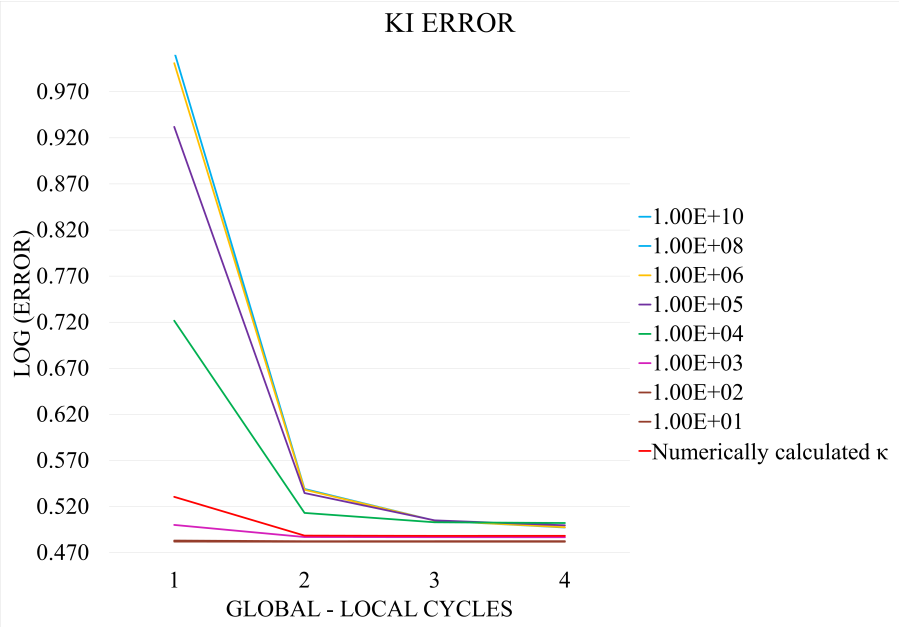


Figure 4. Mixed-Mode Crack – K_I error as a function of κ - Cycles global-local 1 to 4

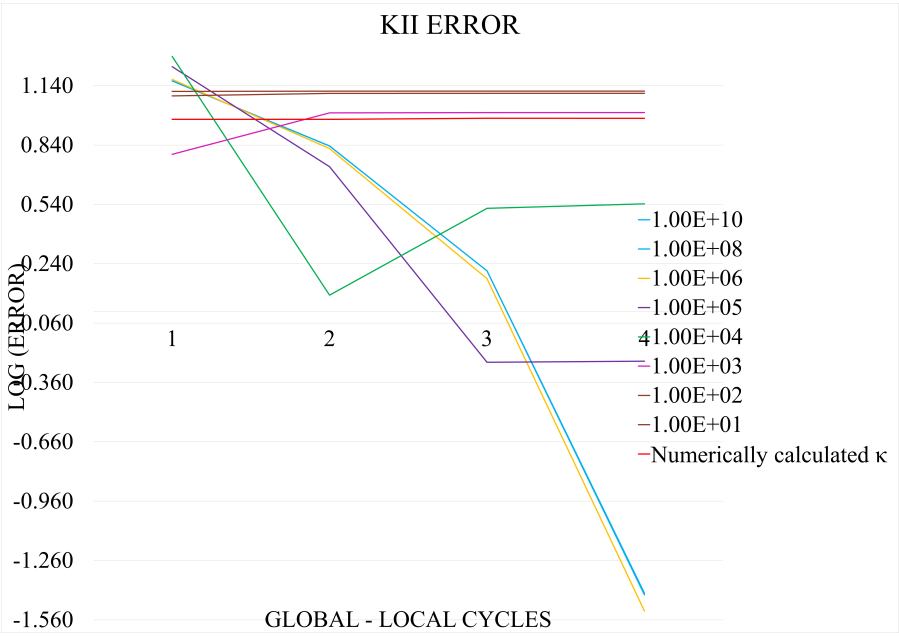


Figure 5. Mixed-Mode Crack – K_{II} error as a function of κ - Cycles global-local 1 to 4

4 Conclusions

Regarding the strain energy, based on the results from the initial global-local cycle, it was observed that higher errors are associated with increased values of κ. A higher κ indicates the use of Dirichlet local boundary conditions, which leads to less accurate enriched global solutions, as noted by Kim et al.[6]. In the subsequent cycles, all analyses demonstrated a decrease in error variation. This trend is expected because the global-local cycles are designed to enhance the accuracy of local boundary data representation. For the problem analyzed across the four global-local cycles, when Cauchy local boundary conditions were applied (here with a recovered stress field), the analyses using the numerically calculated parameter κ showed similar behavior to the other analyses. In fact, the advantage of using a numerically calculated value for κ is not related to improving the enriched global solution, but rather to avoiding reliance on an empirical equation.

Evaluating the stress intensity factors K_I and K_{II}, the analyses using the numerically calculated parameter κ also showed similar behavior to the other analyses using Cauchy local boundary conditions, validating the

mathematical approach.

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