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Geometric nonlinear analysis of space frames constituted of nonprismatic Timoshenko-like elements referred to noncentroidal and nonprincipal axes

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Abstract. This paper presents a geometric nonlinear space frame formulation for nonprismatic Timoshenko-like elements with heterogeneous cross sections. The kinematics of deformation of the element and the cross-sectional constitutive relationships are referenced to an arbitrary reference axis. The element axis may be any simple straight-line segment intersecting the element's cross-sections at any position, regardless of the position of their centroid. Such a noncentroidal and nonprincipal axis requires a consistent consideration of the interaction between axial and flexural effects. The formulation uses a flexibility-type method based on the principle of virtual forces to obtain the structural property coefficients. This method avoids solving the complex differential equations that govern the nonprismatic space frame problem. The Euler-Rodriguez's finite rotation transformation is adopted to determine the natural rotations associated with the nodes of the frame elements along the nonlinear incremental-iterative process. Boundary integrals are adopted to compute the cross-sectional rigidities, which facilitates the modeling of heterogeneous cross sections of complex shapes. All resulting integrals are evaluated using low-order Gauss-Legendre quadrature (3-4 points). The proposed formulation is validated by comparing the nonlinear equilibrium paths, stresses, and internal forces with the ones determined by using highly refined 3D ANSYS models.

Keywords: Geometric nonlinear analysis, nonprismatic space frame element, Timoshenko beam theory, noncentroidal and nonprincipal axis.

1 Introduction

To reduce material consumption, engineers have commonly designed framed structures having long clear spans by adopting shape-optimized, nonprismatic members. In fact, the use of this type of members has not been restricted to reinforced-concrete structures, where the maximization of the strength and stiffness per unit weight is very relevant. Today's evolution in steel technology has made indeed possible the manufacture of custom-designed nonprismatic steel elements. Thus, nonprismatic members are nowadays more commonly seen in Metal Building Frames.

In recent years, many researchers [1–3] investigated nonprismatic beams with curved centroidal axes. Such a nontrivial characteristic, along with varying element rigidity, may raise difficulties for their geometric nonlinear analysis, particularly when formulations based on centroidal axis are employed. Consequently, studying the geometric nonlinear behavior of composite nonprismatic elements with generic cross sections has motivated the development of new effective approaches to describe the equilibrium trajectory of structural systems containing such elements [4–7].

de Araújo and Ribeiro [5] derived exact expressions for geometric and elastic stiffness coefficients of a planar frame element referenced to the centroidal axis. The formulation was based on the Unit Load Method, a particularized form of the Principle of Virtual Forces (PVF). The cross-sectional rigidities were evaluated using boundary integrals. Subsequently, de Araújo et al. [6] extended this work to incorporate constitutive relations for simulating linear semi-rigid connections between elements.

Pillon et al. [8] utilized the Timoshenko Shape Functions (TSFs) presented by de Araújo and Ribeiro [5] to determine the mass and damping matrices of a nonprismatic planar frame element. The authors investigated the dynamic response of a structure subjected to seismic excitation represented by synthetic accelerograms.

Recently, de Araújo and Mageveske [3] expanded the formulation previously introduced by de Araújo and Ribeiro [5] to elements referenced to any noncentroidal axis. The interaction between normal and flexural forces

has been included. This formulation allowed modeling elements with curved centroidal axis, which are usual in nonprismatic structures. Additionally, novel expressions for the TSFs were presented.

This paper aims at expanding the formulation proposed by de Araújo and Mageveske [3] to perform geometrical nonlinear analysis of heterogeneous nonprismatic space frame elements with cross sections of any shape. These elements, described by the Timoshenko Beam Theory, will be referenced to noncentroidal and nonprincipal axis. The exact expressions for the structural-property coefficients and the equivalent nodal load vector will be deduced based on the PVF.

To incorporate both primary (uniform) and higher-order (nonuniform) torsional effects, the method proposed by de Araújo et al. [13] will be adopted. Finally, to validate the proposed formulation, a comparison of results will be presented with a 3D ANSYS [14] simulation.

2 Timoshenko Beam Theory

The approach presented in [5] and [16] will be here extended to space frames. Exact expressions for evaluating structural-property coefficients will be deduced for a space frame element referenced to a noncentroidal and nonprincipal axis. Cross-sectional rigidity values will be evaluated by converting domain integrals into boundary integrals.

2.1 Kinematics of deformation

The displacement field of a space frame element may be described by three translational and three rotational functions. According to the kinematics of deformation of the Timoshenko Beam Theory, based on these functions, the normal displacement $u_1({}^1x_1, \mathbf{x})$ of an arbitrary cross-sectional point $\mathbf{x} \equiv {}^1x_2, {}^1x_3$, referred to any noncentroidal axis, is given by

$$u_1({}^1x_1, \mathbf{x}) = u({}^1x_1) - {}^1x_2 \cdot \beta_3({}^1x_1) + {}^1x_3 \cdot \beta_2({}^1x_1) + \theta_t' \psi(\mathbf{x}), \tag{1}$$

where $u({}^1x_1)$ is the axial component, ${}^1x_2$ and ${}^1x_3$ the cross-sectional location of the point, $\beta_2({}^1x_1)$ and $\beta_3({}^1x_1)$ the bending rotations, $w_2({}^1x_1)$ and $w_3({}^1x_1)$ the vertical components on the respective axis, θ_t' the torsional curvature, and $\psi(\mathbf{x})$ the warping function. Notice that, the superscript indicate the deformed configuration which the variable is referred to, i.e., the C_1 (last determined) configuration [17]. The updated Lagrangian formulation is then adopted to describe the body motion along the incremental-iterative process. Thus, the linear ${}_1e_{ij}$ and nonlinear ${}_1\eta_{ij}$ parts of the updated Green-Lagrange strain tensor can be defined as follows:

$$ {}_1e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial {}^1x_j} + \frac{\partial u_j}{\partial {}^1x_i}\right) \tag{2}$$

$$ {}_1\eta_{ij} = \frac{1}{2}\frac{\partial u_k}{\partial {}^1x_i}\frac{\partial u_k}{\partial {}^1x_j},\ i = j = 1, 2, 3. \tag{3}$$

By considering the relationship of Equation (1) and applying the Generalized Hooke's Law ($\sigma_{11}({}^1x_1, \mathbf{x}) = E({}^1x_1, \mathbf{x}) \cdot u_1'({}^1x_1, \mathbf{x})$) to the isotropic materials composing the cross sections, we derive the following constitutive relationship:

$$\begin{Bmatrix} {}_1N_1({}^1x_1) \\ {}_1M_2({}^1x_1) \\ {}_1M_3({}^1x_1) \end{Bmatrix} = \underbrace{\begin{bmatrix} {}_1k_a({}^1x_1) & {}_1k_{st2}({}^1x_1) & -{}_1k_{st3}({}^1x_1) \\ {}_1k_{st2}({}^1x_1) & {}_1k_{b2}({}^1x_1) & -{}_1k_{23}({}^1x_1) \\ -{}_1k_{st3}({}^1x_1) & -{}_1k_{23}({}^1x_1) & {}_1k_{b3}({}^1x_1) \end{bmatrix}}_{\mathbf{K}} \begin{Bmatrix} u_1'({}^1x_1, \mathbf{x}) \\ \beta_2'({}^1x_1, \mathbf{x}) \\ \beta_3'({}^1x_1, \mathbf{x}) \end{Bmatrix}, \tag{4}$$

where ${}_1N_1({}^1x_1)$, ${}_1M_2({}^1x_1)$ and ${}_1M_3({}^1x_1)$ are the internal forces, and ${}_1k_a({}^1x_1), {}_1k_{sti}({}^1x_1)$, ${}_1k_{ii}({}^1x_1)$, and

${}_1k_{ij}({}^1x_1), i, j = 2, 3$ denotes the cross-sectional rigidity values, given by:

$$
\begin{aligned}
{}_1k_a({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} d\Omega = \int_{\Gamma_m} -{}^1x_2 \cdot n_2 \, d\Gamma \\
{}_1k_{st2}({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} {}^1x_3 d\Omega = \sum_{m=1}^{\text{nmat}} E_m \int_{\Gamma_m} -{}^1x_3 \cdot {}^1x_2 \cdot n_2 \, d\Gamma \\
{}_1k_{st3}({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} {}^1x_2 d\Omega = \sum_{m=1}^{\text{nmat}} E_m \int_{\Gamma_m} {}^1x_2 \cdot {}^1x_3^2 \cdot n_3 \, d\Gamma \\
{}_1k_{b2}({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} {}^1x_3^2 d\Omega = \sum_{m=1}^{\text{nmat}} E_m \int_{\Gamma_m} -{}^1x_2 \cdot {}^1x_3^2 \cdot n_2 \, d\Gamma \\
{}_1k_{b3}({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} {}^1x_2^2 d\Omega = \sum_{m=1}^{\text{nmat}} E_m \int_{\Gamma_m} {}^1x_3 \cdot {}^1x_2^2 \cdot n_3 \, d\Gamma \\
{}_1k_{23}({}^1x_1) =_1 k_{32}({}^1x_1) &= \sum_{m=1}^{\text{nmat}} E_m \int_{\Omega_m} {}^1x_2 \cdot {}^1x_3 d\Omega = \sum_{m=1}^{\text{nmat}} E_m \int_{\Gamma_m} 2 \cdot {}^1x_1^2 \cdot {}^1x_3 \cdot n_2 \, d\Gamma
\end{aligned}
\tag{5}
$$

Note that rigidity values for heterogeneous cross sections may be expressed using boundary integrals [3]. In addition, the system expressed by Equation (4) indicates the couple between axial and flexural deformations by the presence of ${}_1k_{sti}({}^1x_1, \mathbf{x}), i = 2, 3$. Consequently, this formulation does not require the knowledge about the cross-sectional principal axes, since the coefficients ${}_1k_{23}({}^1x_1, \mathbf{x})$ are considered. The shear deformations are expressed by $w_i'({}^1x_1, \mathbf{x}) = {}_1Q_i({}^1x_1)/{}_1k_{si}({}^1x_1), i = 2, 3$.

2.2 Uniform Torsion

The procedure for determining primary (Saint-Venant) torsion is based on the strategy proposed by de Araújo et al. [13]. In [13], a generalized formulation was proposed to describe nonuniform torsion in nonprismatic heterogeneous elements. The authors developed sets of differential equations to describe the torsional phenomena in elements with variable rigidity of polynomial orders (nonprismatic). At the element level, a weighted residuals strategy is adopted for solving the corresponding differential equations. At the local (cross section) level, boundary value problems (BVPs) are solved using the Boundary Element Method (BEM).

These computational techniques allow the determination of the relevant physical quantities in the analysis of structural elements subject to torsion, such as torsional stiffness, warping stiffness, and the shear and normal stress fields.

2.3 Linear stiffness coefficients

For a nonprismatic space frame element referred to any nonprincipal and noncentroidal axis, the Unit Load Method gives:

$$
\begin{aligned}
\bar{f}_j \cdot u_{ji} &= \int_0^{{}^1l} \bar{N}_{1j} \, du_{1i} + \int_0^{{}^1l} \bar{T}_{1j} \, d\theta_t + \int_0^{{}^1l} \bar{M}_{2j} \, d\beta_{2i} + \int_0^{{}^1l} \bar{M}_{3j} \, d\beta_{3i} \\
&\quad + \int_0^{{}^1l} \bar{Q}_{2j} \, dw_{2i} + \int_0^{{}^1l} \bar{Q}_{3j} \, dw_{3i} + u_{jq},
\end{aligned}
\tag{6}
$$

where $\bar{f}_j{=}1$ represents the nodal displacements u_{ij} concerning the deformation mode $i{=}j{=}1, \ldots, 12$ (see de Araújo and Mageveske [3] for details); $\bar{N}_{1j}$, $\bar{T}_{1j}$, $\bar{M}_{2j}$, $\bar{M}_{3j}$, $\bar{Q}_{2j}$, and $\bar{Q}_{3j}$ are the generalized stresses; and, u_{jq} are the nodal displacements due to any loading field ${}^1\mathbf{q} = [{}^1q_1, {}^1q_2, {}^1q_3]^{\mathrm{T}}$. Consequently, from Equation (4), the

linear expressions for the PVF based on the Updated Lagrangian formulation may be written as follows:

$$\begin{aligned} u_{ji} &= \int_0^l \frac{N_{1i}({}^1x_1)\cdot\tilde{N}_{1j}({}^1x_1)}{|\mathbf{K}|({}^1x_1)}\, d^1x_1 + \int_0^l \frac{T_{1i}({}^1x_1)\cdot\tilde{T}_{1j}({}^1x_1)}{k_t({}^1x_1)}\, d^1x_1 \\ &+ \int_0^l \frac{M_{2i}({}^1x_1)\cdot\tilde{M}_{2j}({}^1x_1)}{|\mathbf{K}|({}^1x_1)}\, d^1x_1 + \int_0^l \frac{M_{3i}({}^1x_1)\cdot\tilde{M}_{3j}({}^1x_1)}{|\mathbf{K}|({}^1x_1)}\, d^1x_1 \\ &+ \int_0^l \frac{Q_{2i}({}^1x_1)\cdot\tilde{Q}_{2j}({}^1x_1)}{k_{s2}({}^1x_1)}\, d^1x_1 + \int_0^l \frac{Q_{3i}({}^1x_1)\cdot\tilde{Q}_{3j}({}^1x_1)}{k_{s3}({}^1x_1)}\, d^1x_1 + u_{jq}. \end{aligned} \tag{7}$$

The flexibility coefficients can be obtained rewritten the Equation (7) in the following algebraic form:

$$\sum_{k=k_0}^{k_1} a_{jk}\cdot{}_1f_{ki} = u_{ji} - \sum_{k=k_0}^{k_1} g_{jki}\cdot{}^1f_k - u_{jq},\ k_0 = [(m-1)\cdot\text{ndofn}]+1,\ k_1 = m\cdot\text{ndofn}. \tag{8}$$

where, $j,i = 1,\ldots,6\,(m=1)$, $j,i = 7,\ldots,12\,(m=2)$, $i = q$, and 'ndofn' is the number of the degree of freedom. To obtain the expression for the elastic stiffness coefficients ${}_1k_{kj} = {}_1f_{ki}$ the nonlinear and element loading terms should be neglected, i.e. $\sum_{k=k_0}^{k_1} g_{jki}\cdot{}^1f_k = u_{jq} = 0$.

Details regarding the derivation of the aforementioned formulation are presented in [3, 6] for both linear and geometric nonlinear terms. The strategy proposed in [13] will be utilized to include both primary and higher-order torsional effects. In [18] a finite element formulation for space frame elements referenced to the centroidal axis and nonprincipal axis is presented. This formulation will be expanded to include noncentroidal axis.

2.4 Second-Order geometric effects

By definition, geometric stiffness results from changes in the configuration equilibrium of a structural system with no increment in the external work, i.e., ${}^2_1\text{W}_{\text{ext}} - {}^1_1\text{W}_{\text{ext}} = 0$. Thus, the Principle of Virtual Displacements [17] give us

$${}^2_1\text{W}_{\text{ext}} - {}^1_1\text{W}_{\text{ext}} = \int_{{}_1\Omega} {}_1C_{mnkl}\cdot{}_1e_{kl}\cdot\delta({}_1e_{mn})d^1\Omega + \int_{{}_1\Omega} {}^1\tau_{mn}\cdot\delta({}_1\eta_{mn})d^1\Omega = 0 \tag{9}$$

where ${}_1C_{mnkl}$ is the constitutive matrix for a isotropic material and ${}_1e_{kl}$ and ${}_1\eta_{mn}$ are the linear and nonlinear parts of the Green-Lagrange increment tensor - Equation (2). In this paper, following the methodology presented in [16] for plane frames, we use relationship 9 to derive the expressions for the geometric nonlinear internal-force increments for space frames when it moves from C1 to C2. Including these terms in Equation (8), we obtain

$$\begin{aligned} \sum_{k=k_0}^{k_1} a_{jk}\cdot{}_1f_{ki} &= \int_0^{{}^1l} N^g_{1j}\, du_{1i} + \int_0^{{}^1l} T^g_{1j}\, d\theta_t + \int_0^{{}^1l} M^g_{2j}\, d\beta_{2i} + \int_0^{{}^1l} M^g_{3j}\, d\beta_{3i} \\ &+ \int_0^{{}^1l} Q^g_{2j}\, dw_{2i} + \int_0^{{}^1l} Q^g_{3j}\, dw_{3i}. \end{aligned} \tag{10}$$

Notice that the right-hand side of the Equation (10) represents the g_{jki} terms presented on Equation (8). Then, once the g_{jki} terms are known, one may obtain the geometric stiffness coefficients by solving Equation (10) (see [6] and [3] for details).

2.5 Euler-Rodriguez formula

To update the nodal coordinates of the space frame element we adopted the concept of finite rotation presented by Euler-Rodriguez [17]. The transformation of a vector $\mathbf{r}$ to a vector $\mathbf{r}'$ due to a rigid rotation ϕ about OS axis are described by

$$\mathbf{r}' = \mathbf{r}\cos\phi + (\mathbf{n}\times\mathbf{r})\sin\phi + (1-\cos\phi)(\mathbf{n}\cdot\mathbf{r})\mathbf{n}, \tag{11}$$

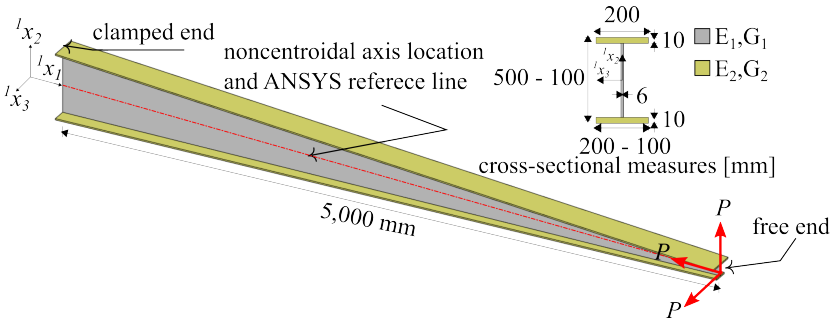
where $\mathbf{n}$ is the unit vector along the axis defined by the segment OS. This formula is used to determine the natural rotations of the frame element end nodes during the incremental step from C_1 to C_2 configuration. Once the natural deformations are known, we can eliminate the rigid body motion from the incremental displacement vector to determine the element internal forces.

2.6 Nonlinear solver

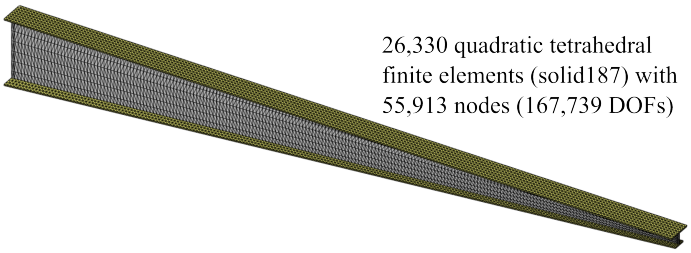
This study adopted the Generalized Displacement Control Method (GDCM) with the generalized stiffness parameter (GSP) as a load increment [19].

3 Monosymmetric heterogeneous I-beam with variable web and bottom flange

In this section is analyzed a nonprismatic, monosymmetric I-beam with the web made of structural steel (material 1) and flanges made of an aluminum alloy (material 2) - Figure 1a. Material 1 has Young's modulus $E_1 = 200$ GPa and a shear modulus $G_1 = 76.9$ GPa, while material 2 has $E_2 = 71$ GPa and $G_2 = 26.7$ GPa. The cross-sectional geometry is assumed to vary linearly. The corresponding measures are presented in Figure 1a. The obtained results were validated using highly accurate 3D numerical simulation conducted with the commercial finite element package ANSYS - Figure 1b. The arc-length method was employed within the minimum arc-length of 10^{-7} and a maximum arc-length of 1 with 500 load steps.



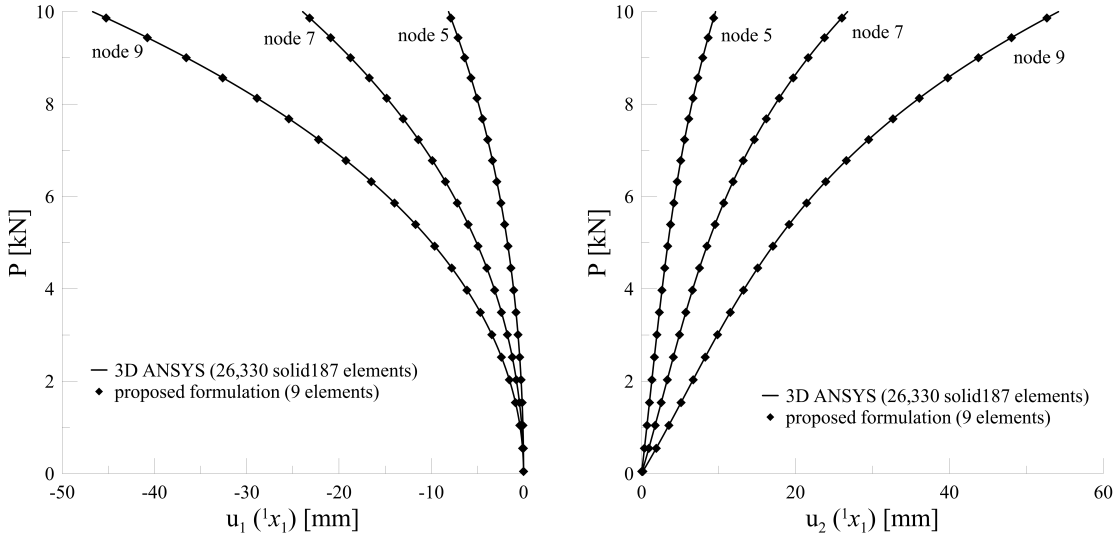
(a) I-beam geometry and cross-sectional description



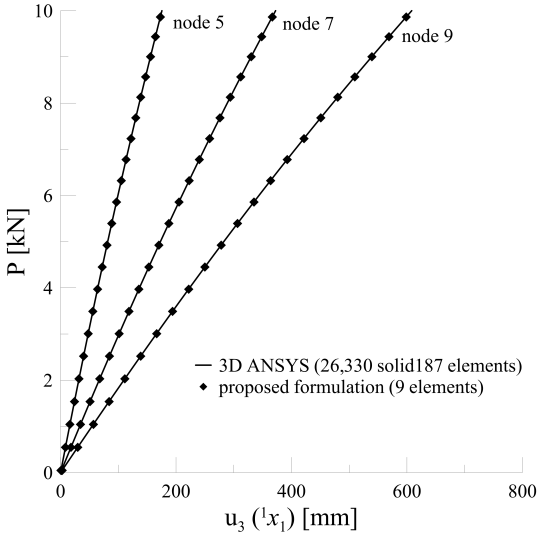
(b) 3D ANSYS solid 187 finite element mesh

Figure 1. Monosymmetric I-beam with heterogeneous cross section

The axis location remains at the same position at half the height of the beam. The loading case adopted in the study was $P = 10$ kN. The nonlinear problem was solved with the following parameters: 9 finite space frame elements, initial load λ_0^0=10^{-3} and force and displacement tolerance equal to 10^{-3}. The number of load steps was 1,237 with 3-4 iterations per load step. Finally, Figure 2a, 2b, and 2c portrait the results, where are observed excellent agreement with 3D ANSYS results.



(a) $P \times u_1(^1x_1)$ (b) $P \times u_2(^1x_1)$



(c) $P \times u_3(^1x_1)$

Figure 2. Equilibrium trajectories

4 Conclusions

This paper proposed an extension of the method previously investigated by de Araújo et al. [16], incorporating the coupling between axial and flexural effects. Structural-property coefficients were derived based on the PVF in the form of the Unit Load Method. To validate the robustness of the proposed formulation, a monosymetric, heterogeneous I-beam with a curvilinear centroidal axis was analyzed. The results obtained from this study demonstrated excellent agreement with 3D ANSYS simulations.

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