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# Study of cable vibration by large deformations using Rayleigh and Runge-Kutta methods

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**Abstract.** The hypothesis of large deformations is a working condition assumed for restricted structural systems, and it finds specific applications in engineering practice. Problems involving cables are found in this context. The study of cables is a relevant topic for global industry because they are essential elements for modern life. In particular, the study of their vibration arouses special interest because cables are used, among numerous applications, for the transmission of electrical energy. In this condition, they are subject to environmental excitations, which can represent a source of resonance. The vibration of a cable always occurs in the deformed configuration, therefore under the influence of the loadings. The first is its own weight, the second is an axial force that allows the system to exist and keeps the cable tensioned in its working condition. The tensile force modifies the stiffness of the system and its natural vibration frequencies. In the context of what has been described, the present work aims to study the vibration of a structural system formed by a cable, assuming the hypothesis of large deformations. The adopted model will be that of a simply supported beam at its ends and subjected to both loads, one uniformly distributed due to the self-weight of the structural element, and a tensile force, which will be applied in conjunction with the first one, under the condition assumed for the deformed state of the cable. The analytical method to be employed for determining the vibration frequency is based on the Rayleigh method, for which the shape function will be obtained numerically by solving the exact elastic line of the beam model. For this, the differential equations that characterize that hypothesis will be solved using the Runge-Kutta method for the boundary conditions of the problem, in the Mathcad environment.

**Keywords:** cable, natural frequency, exact elastic line, Rayleigh method, Runge-Kutta method.

## 1 Introduction

Cable studies, which include displacements and frequencies, must generally be performed assuming large deformations due to the geometric and material characteristics of these systems. Cables are vitally important elements in global and Brazilian industry due to their numerous applications, including electrical power transmission. Cables of transmission lines are located at climatic conditions capable of mobilizing their vibration modes. Dua et al. [1] emphasize that the response of cables to wind effects is quite complex due to flow turbulence and the system's inherent geometric and material nonlinearities. Oliveira et al. [2] reported a substantial number of transmission line accidents that have occurred at wind speeds below design, which may be related to these nonlinearities. Onunka and Ojo [3] used a typical beam element to represent transmission line cables when calculating their natural frequencies.

It is important to emphasize that cables are extremely thin and highly flexible structural elements, conditions inherent to their own nature. For this reason, the equilibrium of cable systems can only be found in their deformed configuration, even when under tension. The magnitude of the tensile force to which they are subjected modifies their natural frequencies because the axial force mobilizes the portion of the geometric stiffness that is included in their total stiffness (Wahrhaftig [4]-[5]). Therefore, there are two components of cable stiffness. The first, called flexural stiffness, depends entirely on the material's modulus of elasticity and the inertia of the cross-

section. The second, called geometric stiffness, is a function of the acting normal force. The second reflects on the first one because it contributes to a greater or lesser reluctance to lateral movement.

There are also two Elastica of a cable. The first one is approximated, calculated under linear geometric conditions. The second one is exact, determined under the assumption of large deformations. It is presumed that assuming one or other leads the structural analyses to substantially different results. This is the central objective of this work. In the simulations performed here to evaluate the results obtained by adopting one or the other, the elastic line equations are solved numerically using the Runge-Kutta method. The natural frequency is then determined using the Rayleigh method [6]. To define the deformed configuration of the system, percentage values of the steel's modulus of elasticity are adopted, which vary between 70 and 460%. The lower limit was defined as the lowest possible value capable of making the numerical solution feasible. The upper limit was established to drive both deformed configurations to the same value, with an arbitrarily assumed tolerance. Both methods, Runge-Kutta and Rayleigh, are implemented in Mathcad (v.13).

# 2 Problem description

The present problem concerns the determination of the natural frequency of a cable simulated as a beam, as shown in Fig. 1. Initially, the cable's elastic line is calculated under the assumptions of small and large deformations. Then, the system's natural frequencies are determined for these two assumptions. Consider the model proposed in Fig. 1, representing a simply supported beam *AB* of length *L*, with flexural stiffness *EI*, tensioned by a force *P* and subjected to a uniformly distributed load *q* originating from its own weight.

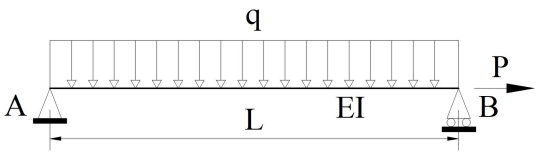



Figure 1. Modelling the problem

## 2.1 Elastica definition

The elastic line equation describes the position that the axis of the structural element occupies when bent, and can be obtained based on a differential element *dx* located between two adjacent sections α and β forming an opening $d\theta$, as indicated in Fig. 2. The tangent line to the elastic line of the beam passing through section α forms an angle $\theta$ with the horizontal. The height of the deformed beam relative to the differential element is *dy*.

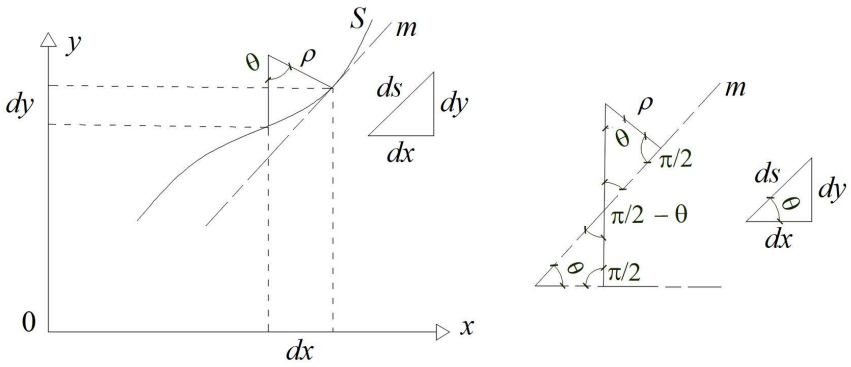



Figure 2. Elastica definition

By the trigonometric relation of the right triangle formed by the differentials, *ds*, *dx* e *dy*, we have that:

$$ds^2 = dx^2 + dy^2 \tag{1}$$

Dividing all terms of eq. (1) by $dx^2$, we obtain:

$$\frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2} \rightarrow ds = dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2}$$

The opening $\theta$ can be defined by a straight line tangent to the curve, $m$, so that:

$$tg\theta = \frac{dy}{dx} \tag{3}$$

The derivative of eq. (3 with respect to $x$ is found by doing:

$$\frac{d}{dx}\left(tg\theta\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right) \rightarrow \frac{d}{d\theta}\left(tg\theta\right)\frac{d\theta}{dx} = \left(1 + tg^2\theta\right)\frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \tag{4}$$

Therefore,

$$\left(1 + tg^2\theta\right)\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \tag{5}$$

Taking eq. (3) into (5), we have:

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \;\rightarrow\; d\theta = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}dx \tag{6}$$

In turn, the length of an arc of a circle is defined by $ds = \rho d\theta$, which leads to:

$$\rho = \frac{dx\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}{\dfrac{d^2y}{dx^2}dx} \;\rightarrow\; \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{7}$$

with

$$\frac{1}{\rho} = \frac{M}{EI} \rightarrow \frac{M}{EI} = -\frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}}} \tag{8}$$

The Eq. (8) is known as the “exact equation” of the elastic line. If the higher-order infinity in the denominator is neglected, this equation becomes the one that defines the approximate elastic, eq. (9).

$$\frac{M}{EI} = -\frac{d^2y}{dx^2} \tag{9}$$

### 2.2 Runge-Kutta Method

The fourth-order Runge-Kutta method is among the most widely applied in engineering. It consists of a numerical technique that enables the solution of first-order ordinary differential equations (ODEs), state Gilat and

Subramaniam [7]. Ma et al. [8] point out that Runge-Kutta is commonly used in numerical simulations because it reduces computational effort, making the simulation simpler and easier to execute. The method is represented by the following equation:

$$y_{i+1} = y_i + \frac{1}{6}\left(K_1 + 2K_2 + 2K_3 + K_4\right)h \tag{10}$$

in which the constants $K_1$, $K_2$, $K_3$, $K_4$ are determined by:

$$K_1 = f\left(x_i, y_i\right) \tag{11}$$

$$K_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_1 h\right) \tag{12}$$

$$K_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}K_2 h\right) \tag{13}$$

$$K_4 = f\left(x_i + h, y_i + K_3 h\right) \tag{14}$$

### 2.3 Rayleigh Method

The equations recommended by Rayleigh for computing the natural frequency of a mechanical system are based on the total stiffness of the system, $K$, eq. (15), which is composed of two components: flexural stiffness, $k_F$, and geometric stiffness, $k_G$, eqs. (16) and (17), respectively. The first depends fundamentally on the material properties and the second on the applied axial force, point out Wahrhaftig et al. [9].

$$K = k_F + k_G \tag{15}$$

$$k_F = \int_0^L EI\left[z''\left(x\right)\right]^2 dx \tag{16}$$

$$k_G = P\int_0^L \left[z'\left(x\right)\right]^2 dx \tag{17}$$

The total mass of the system $M$ is defined by eq. (18), with $\overline{m} = q/g$ being the mass per unit length originating from the structural element's self-weight.

$$M = \int_0^L \overline{m}\left[z'\left(x\right)\right]^2 dx \tag{18}$$

In the previous equations, $z(x)$ is the elastic frequency of the system, with $z'(x)$ and $z''(x)$ representing the first and second derivatives. Thus, the natural frequency can be determined by:

$$\omega = \sqrt{\frac{K}{M}} \text{ (in rad/s) and } f = \frac{\omega}{2\pi} \text{ (in Hz)} \tag{19}$$

## 3 Numerical simulation, results and discussion

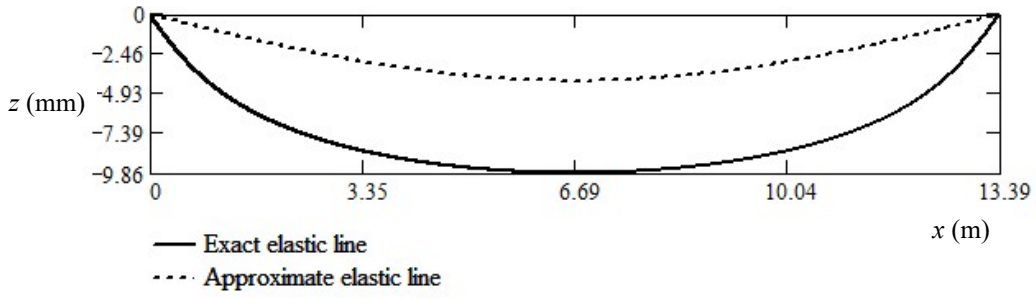
The numerical simulation was conducted to evaluate the structural response considering different ratios $n$ between the modulus of elasticity $E$ of the material and that of the steel (210 GPa), ranging from 70 to 460%. The cable has a diameter $d$ of 0.0183 m, therefore an inertia $I$ equal to $5.51 \times 10^{-9}$ m$^4$, a length $L$ of 13.385 m, and a specific weight $\gamma$ de 30301.133 N/m$^3$. The tensile force $P$ is 15860 N, and the acceleration of gravity g has a value

equal to 9.806650 m/s$^2$. The numerical solutions of the Rugen-Kutta and Rayleigh methods were implemented in Matchcad program. In this environment, the Runge Kutta method obeys the "Odesolve" command, which solves eqs (8) and (9) after imposing the problem's boundary conditions, that is, zero displacement and rotation, on the support for $x = 0$, and at midspan for $x = L/2$, respectively. The simulation was run until the difference between the frequency calculated with the approximate elastic method and that calculated with the exact elastic method was less than 0.005%, an arbitrarily defined limit.

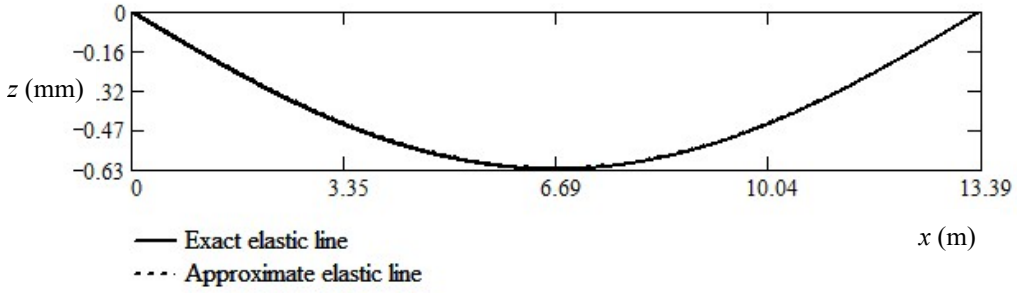
The results for mid-span displacements, by the approximate elastic $z_{ap}$ and exact $z_{ex}$, as well as for the frequencies by the approximate and exact elastic $f_{ap}$ and $f_{ex}$ are presented in Table 1. The deformations by the approximate and exact hypotheses can be seen in Figs. 3(a), for $E = 70\%$, and 3(b), for $E = 460\%$. The comparison between the displacements obtained by both hypotheses can be seen in Fig. 4, and for the frequency, in Fig. 5.

Table 1. Results for displacement and frequencies

| $n$ | $z_{ap}$ (mm) | $z_{ex}$(mm) | $(1-z_{ap}/z_{ex})$ % | $f_{ap}$ (Hz) | $f_{ex}$ (Hz) | $(1-f_{ap}/f_{ex})$ % | $(1-f_{ap(n+1)}/f_{ap(n)})$ % | $(1-f_{ex(n+1)}/f_{ex(n)})$ % |
|---|---|---|---|---|---|---|---|---|
| 70% | 4.1160 | 9.8597 | 58.254 | 5.2261 | 5.8956 | 11.356 | - | - |
| 75% | 3.8416 | 6.5256 | 41.131 | 5.2266 | 5.4024 | 3.254 | -0.010 | 8.365 |
| 80% | 3.6015 | 5.3800 | 33.058 | 5.2272 | 5.3196 | 1.738 | -0.010 | 1.533 |
| 85% | 3.3896 | 4.6872 | 27.684 | 5.2277 | 5.2858 | 1.100 | -0.010 | 0.635 |
| 90% | 3.2013 | 4.1975 | 23.733 | 5.2282 | 5.2683 | 0.762 | -0.010 | 0.332 |
| 95% | 3.0328 | 3.8233 | 20.676 | 5.2287 | 5.2581 | 0.558 | -0.010 | 0.195 |
| 100% | 2.8812 | 3.5235 | 18.229 | 5.2293 | 5.2516 | 0.425 | -0.010 | 0.123 |
| 110% | 2.6193 | 3.0657 | 14.562 | 5.2303 | 5.2444 | 0.269 | -0.020 | 0.137 |
| 120% | 2.4010 | 2.7268 | 11.950 | 5.2313 | 5.2410 | 0.184 | -0.020 | 0.065 |
| 130% | 2.2163 | 2.4627 | 10.005 | 5.2324 | 5.2394 | 0.134 | -0.020 | 0.031 |
| 140% | 2.0580 | 2.2496 | 8.518 | 5.2334 | 5.2387 | 0.101 | -0.020 | 0.013 |
| 150% | 1.9208 | 2.0730 | 7.345 | 5.2345 | 5.2386 | 0.079 | -0.020 | 0.002 |
| 160% | 1.8007 | 1.9239 | 6.404 | 5.2355 | 5.2388 | 0.063 | -0.020 | -0.004 |
| 190% | 1.5164 | 1.5873 | 4.467 | 5.2387 | 5.2406 | 0.037 | -0.060 | -0.033 |
| 220% | 1.3096 | 1.3543 | 3.301 | 5.2418 | 5.2431 | 0.024 | -0.060 | -0.047 |
| 250% | 1.1525 | 1.1825 | 2.537 | 5.2449 | 5.2458 | 0.017 | -0.060 | -0.053 |
| 280% | 1.0290 | 1.0501 | 2.009 | 5.2481 | 5.2487 | 0.013 | -0.060 | -0.055 |
| 310% | 0.9294 | 0.9449 | 1.640 | 5.2512 | 5.2517 | 0.010 | -0.060 | -0.057 |
| 340% | 0.8474 | 0.8591 | 1.362 | 5.2543 | 5.2547 | 0.008 | -0.060 | -0.058 |
| 370% | 0.7787 | 0.7877 | 1.143 | 5.2574 | 5.2578 | 0.007 | -0.059 | -0.058 |
| 400% | 0.7203 | 0.7274 | 0.976 | 5.2606 | 5.2609 | 0.006 | -0.059 | -0.058 |
| 430% | 0.6700 | 0.6758 | 0.858 | 5.2637 | 5.2639 | 0.005 | -0.059 | -0.059 |
| 460% | 0.6263 | 0.6310 | 0.745 | 5.2668 | 5.2670 | 0.004 | -0.059 | -0.059 |




(a)




(b)

Figure 3. Elastic lines exact and approximate for $E$ = 70% (a), and $E$ = 460% (b)

It can be observed that for $E$ = 70%, the displacement at the center of the span obtained by the approximated elastic line equation represents 60% of the exact value, decreasing exponentially as the modulus increases, Fig. 4. The structural frequency, in turn, exhibits two distinct behaviors. When the approximated elastic line is used as a shape function, the frequency increases linearly with the modulus of elasticity. When the exact elastic line is used, the structural frequency initially decreases, even with increasing $E$, until the value of 150%, when it begins to increase linearly. The drop in frequency between 70 and 75% exceeds 8%.

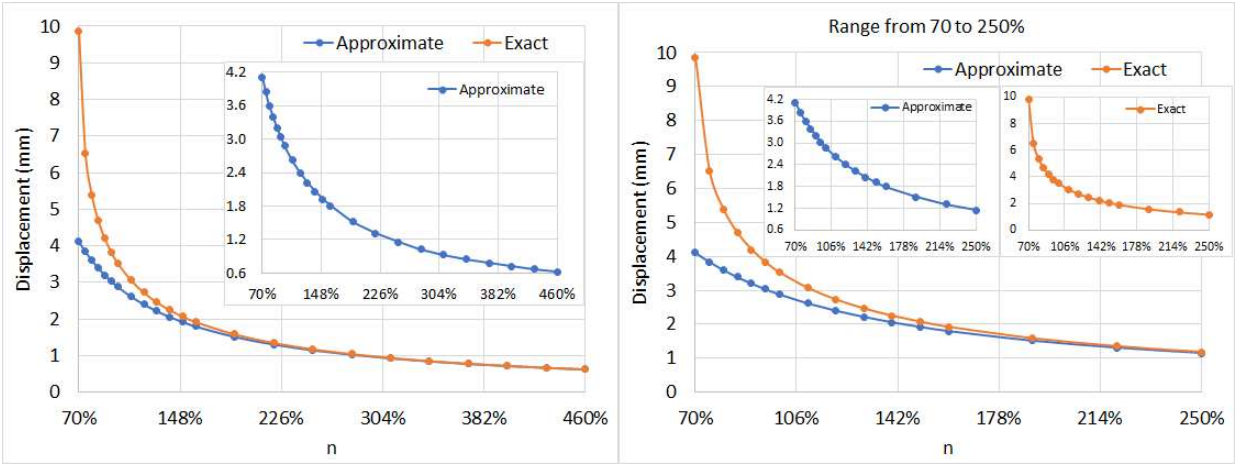



Figure 4. Displacements at $L$/2

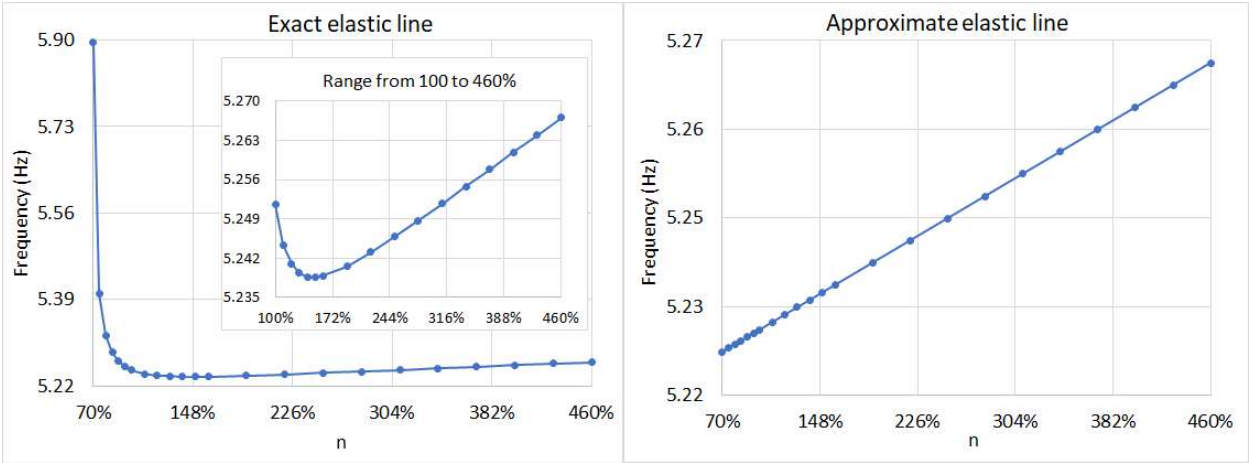



Figure 5. Structural frequencies

# 4 Conclusions

In this work, the natural frequencies of a cable modeled as a beam were evaluated using small and large deformation assumptions. The elastic line equations were solved numerically using the Runge-Kutta method, and the natural frequencies were solved using the Rayleigh method. The steel modulus was used as a reference, assuming variations between 70 and 460% of its value. The following can be concluded:

- The displacements in both assumptions, approximate and exact, always decrease with increasing modulus of elasticity and obeying a potential law.
- The approximate elastic strain at midspan is 60% of the exact value for the lowest modulus of elasticity, with a tendency to converge after 250%.
- The natural frequency calculated using the approximate elastic line assumption increases linearly with the modulus of elasticity.
- The structural frequency calculated using the exact elastic line assumption shows an initial decrease even as the modulus of elasticity increases, up to a value corresponding to 150%. Then, it reverses the trend and presents values that are always increasing linearly.
- The frequency of the nonlinear elastic deformation hypothesis decreases by more than 8% for a variation of approximately 7% in the value of the elastic modulus. The percentage differences then decrease until they are reduced to values with very little variation for the subsequent modulus of elasticity.

Future work involving the use of the finite element method (FEM) is expected.

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**Authorship statement.** The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

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