XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING

# ANALYSIS OF RC BEAMS ON ELASTIC FOUNDATIONS BASED ON ANALYTICAL INTERPOLATION FUNCTIONS

Mateus R. Zanelato[1], Marcos A. C. Rodrigues[1]

[1] *Dept. of Civil Engineering, Federal University of Espírito Santo*
*Fernando Ferrari Avenue, 514, 29075-910, Vitória/ES, Brazil*
mateus.zanelato@edu.ufes.br, marcos.a.rodrigues@ufes.br

**Abstract.** Reinforced concrete beams resting on an elastic basis represent a typical problem of structural foundations. Although analytical solutions exist, the analysis of such structures is generally carried out using the Finite Element Method (FEM) due to the complexity introduced by different types of loading, support conditions and geometries. The classical FEM approach typically requires a refined discretization of the beam to accurately capture the elastic behavior of the supporting soil. Thus, the soil-structure interaction is simulated using springs along the nodes of the discretized model. However, when FEM interpolation functions are derived directly from the governing differential equations, these functions are analytical, and the discretization is not essential. This study employs an Euler-Bernoulli beam element formulation based on analytical interpolation functions considering the soil-structure interaction. This formulation is capable of modeling and simulating reinforced concrete beams with the minimum number of elements required to accurately describe geometry, or variation in soil properties. The numerical application demonstrates the capacity of the element in the design process of reinforced concrete beams on elastic foundation considering just a single element to model the structure.

**Keywords:** RC Beams, elastic foundation, analytical interpolation functions, finite element method, single element.

## 1 Introduction

Civil engineering, in its continuous pursuit of innovation and adaptation to growing technological demands, has increasingly focused on developing efficient and robust solutions for contemporary structural challenges. To address these challenges, structural elements are discretized into smaller sub-elements, allowing for a more accurate prediction of their behavior, as shown by Zienkiewicz et. al [1]. This process is widely supported by computational tools, particularly software based on the Finite Element Method (FEM), extensively used in engineering for modeling and simulating complex structures. FEM solves structural problems through the formulation of differential and integral equations, providing approximate solutions for discretized elements.

Within this context, one of many critical factors involves simulating elastic foundations, where structural discretization is intrinsic to FEM. Typically, as presented by Liu et. al [2], each node is associated with a spring to model the elastic support, and a higher number of elements and nodes leads to a more accurate representation of the foundation. Nonetheless, although refined discretization the accuracy of the results, two main factors must be considered when defining the appropriate number of elements: computational capacity and the engineer's expertise.

Both are shown by Silva and Burgos [3], where the first factor is related to computational capacity, as excessive mesh refinement can require significant processing time, making the analyses computationally expensive, especially for large-scale structures. While the second factor is associated with the engineer's expertise, since the appropriate definition of the mesh depends on the professional's experience in selecting a refinement that balances accuracy and computational efficiency.

Therefore, the present work explores a formulation based on the FEM for analyzing concrete elements resting on elastic foundations, using a single finite element. The method is applied to beams elements resting on

elastic foundations and employs analytical interpolation functions. These functions are derived directly from the analytical solution of the differential equation governing an infinitesimally deformed element on an elastic base. The element stiffness matrix is then obtained using these analytical interpolation functions, enabling the analysis of continuous structural behavior without the need for a finite element discretization.

The study aims to explore this formulation for analyses of reinforced concrete beams resting on elastic foundations using a single element, making advanced structural analysis techniques more accessible to less-experienced professionals, including undergraduate and graduate students and researchers.

# 2 Element Formulation

## 2.1 Euler-Bernoulli Beam Theory

Figure 1 shows an infinitesimal element in its undeformed configuration over an elastic foundation. Based on the equilibrium of internal forces derived from the universal equations of statics, studied by Burgos [4], eq. (1) and (2) are formulated, where $Q$ is the vertical shear force at the cross-section, $M$ is the bending moment at the section, $q$ is the transverse distributed load on the infinitesimal element, $v$ is the transverse displacement, and $k^v$ is the spring coefficient used to simulate the soil's behavior.

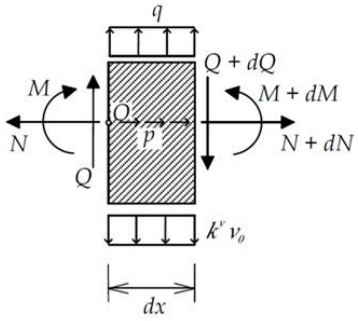



Figure 1. Equilibrium of an undeformed infinitesimal element on an elastic foundation (adapted from Rodrigues [5])

$$\sum F_y = -dQ + q(x)dx - k^v v_0 dx = 0 \therefore \frac{dQ(x)}{dx} = q(x) - k^v v_0(x), \tag{1}$$

$$\sum M_0 = dM - (Q + dQ)dx + q(x)\frac{dx^2}{2} - k^v v_0(x)\frac{dx^2}{2} = 0 \ \therefore \frac{dM(x)}{dx} - k^v v_0(x)\frac{dx^2}{2} = Q(x). \tag{2}$$

By then applying the elastic line equations and the principles of Euler-Bernoulli Beam Theory, which states that the rotation at any point of the element is given by the derivative of the transverse displacement, it is possible to obtain eq. (3), (4) and (5). These equations describe the behavior at any point of the structural element, considering the solution as the response of the homogeneous part of the problem, achieved by setting $q(x) = 0$.

$$v(x) = cosh(ax) \cdot [c_1 \cdot cos(bx) + c_2 \cdot sen(bx)] + senh(ax) \cdot [c_3 \cdot cos(bx) + c_4 \cdot sen(bx)], \tag{3}$$

$$\begin{aligned} \theta(x) = {} & cosh(ax) \cdot (b \cdot c_2 \cdot cos(bx) - b \cdot c_1 \cdot sin(bx)) \\ & + sinh(ax) \cdot (b \cdot c_4 \cdot cos(bx) - b \cdot c_3 \cdot sin(bx)) \\ & + a \cdot cosh(ax) \cdot (c_3 \cdot cos(bx) + c_4 \cdot sin(bx)) \\ & + a \cdot sin\,h(ax) \cdot \big(c_1 \cdot cos(bx) + c_2 \cdot sin(bx)\big), \end{aligned} \tag{4}$$

where,

$$a = b = \sqrt[4]{\frac{k^v}{4EI}} \,. \tag{5}$$

## 2.2 Analytical Interpolation Functions

When using FEM to simulate the analytical behavior of a solid element, its continuous solution is typically obtained by interpolating the discrete nodal displacements. The deformed condition are given by the displacements of the end nodes, as shown in the Fig. 2, wherein $d'_1$ and $d'_4$ represent the horizontal displacements of each node, $d'_2$ and $d'_5$ the vertical displacements, and $d'_3$ and $d'_6$ the rotations.

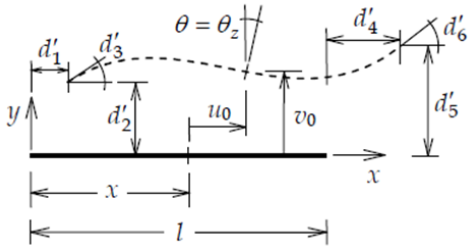



Figure 2. Deformed condition of a single beam (adapted from Martha [6])

The equations governing the behavior of bars through their homogeneous solutions can also be described according to the nodal displacements, using interpolation functions. Equation (6) presents the corresponding matrix formulation, where the position of any point along the continuous element can be described by a matrix of shape functions $[N]$ multiplied by $\{d\}$, a vector composed of the displacements at the initial and final nodes of the element.

$$\begin{Bmatrix} u_0(x) \\ v_0(x) \\ \theta(x) \end{Bmatrix} = \begin{bmatrix} N_1^u(x) & 0 & 0 & N_4^u(x) & 0 & 0 \\ 0 & N_2^v(x) & N_3^v(x) & 0 & N_5^v(x) & N_6^v(x) \\ 0 & N_2^\theta(x) & N_3^\theta(x) & 0 & N_5^\theta(x) & N_6^\theta(x) \end{bmatrix} \cdot \begin{Bmatrix} d'_1 \\ d'_2 \\ d'_3 \\ d'_4 \\ d'_5 \\ d'_6 \end{Bmatrix} \tag{6}$$

To avoid the structural discretization, it is possible to obtain the analytical interpolation functions directly from the homogeneous solution of the differential equation of the problem: an infinitesimal undeformed element on an elastic foundation. So, by using eq. (3) and (4), as well as its boundary conditions, and applying to eq. (6) it's possible to isolate the interpolation functions and obtain the matrix $[N]$, as shown in eq. (7) to (14).

$$\begin{aligned} N_2^v(x) = \quad & -(a^2 \cdot cosh(ax) \cdot cos(bx) + b^2 \cdot cosh(ax) \cdot cos(bx)) - \\ & a^2 \cdot cos\big(b \cdot (2L - x)\big) \cdot cosh(ax) - b^2 \cdot cosh\big(a \cdot (2L - x)\big) \cdot cos(bx) - \\ & \frac{a \cdot b \cdot sinh\big(a \cdot (2L - x)\big) \cdot sin(bx) + a \cdot b \cdot sin\,(b \cdot (2L - x)) \cdot sinh\,(ax))}{a^2 \cdot cos\,(2Lb) + b^2 \cdot cosh(2La) - a^2 - b^2} \end{aligned} \tag{7}$$

$$\begin{aligned} N_2^\theta(x) = & -((a^2 + b^2) \cdot (a \cdot co\,s(bx) \cdot sin\,h(ax) - b \cdot cosh(ax) \cdot sin(bx) - \\ & \frac{a \cdot cos\big(b \cdot (2L - x)\big) \cdot sinh(ax) + b \cdot cosh(a \cdot (2L - x)) \cdot sin(bx))}{a^2 \cdot cos(2Lb) + b^2 \cdot cosh(2La) - a^2 - b^2} \end{aligned} \tag{8}$$

$$N_3^v(x) = -(a\cdot cos\,(bx)\cdot sinh\,(ax) + b\cdot cosh\,(ax)\cdot sin\,(bx) - \frac{a\cdot cos\big(b\cdot(2L-x)\big)\cdot sinh(ax) - b\cdot cosh(a\cdot(2L-x))\cdot sin(bx))}{a^2\cdot cos(2Lb) + b^2\cdot cosh(2La) - a^2 - b^2} \quad (9)$$

$$N_3^\theta(x) = -(a^2\cdot cosh\,(ax)\cdot cos\,(bx) + b^2\cdot cosh\,(ax)\cdot cos\,(bx) - a^2\cdot cos\big(b(2L-x)\big)\cdot cosh(ax) - b^2\cdot cosh\big(a\cdot(2L-x)\big)\cdot cos(bx) + \frac{a\cdot b\cdot sinh\big(a\cdot(2L-x)\big)\cdot sin(bx) - a\cdot b\cdot sin(b\cdot(2L-x))\cdot sinh(ax))}{a^2\cdot cos(2Lb) + b^2\cdot cosh(2La) - a^2 - b^2} \quad (10)$$

$$N_5^v(x) = -(b^2\cdot cos\,(Lb)\cdot sinh\,(La)\cdot cos\,(bx)\cdot sinh\,(ax) - a^2\cdot cosh(La)\cdot sin(Lb)\cdot cosh(ax)\cdot sin(bx) + a^2\cdot sinh(La)\cdot sin(Lb)\cdot sinh(ax)\cdot sin(bx) + b^2\cdot sinh(La)\cdot sin(Lb)\cdot sinh(ax)\cdot sin(bx) + a\cdot b\cdot cosh(La)\cdot sin(Lb)\cdot cos(bx)\cdot sinh(ax) - \frac{a\cdot b\cdot cos(Lb)\cdot sinh(La)\cdot cosh(ax)\cdot sin(bx))}{-a^2\cdot cos(Lb)^2 + a^2 - b^2\cdot cosh(La)^2 + b^2} \quad (11)$$

$$N_5^\theta(x) = (a^2+b^2)\cdot(a\cdot cosh(La)\cdot sin(Lb)\cdot sinh(ax)\cdot sin(bx) - a\cdot sinh(La)\cdot sin(Lb)\cdot cosh(ax)\cdot sin(bx) + b\cdot cos(Lb)\cdot sinh(La)\cdot sinh(ax)\cdot sin(bx) - \frac{b\cdot sinh(La)\cdot sin(L\cdot b\cdot cos(bx)\cdot sinh(ax))}{-a^2\cdot cos(Lb)^2 + a^2 - b^2\cdot cosh(La)^2 + b^2} \quad (12)$$

$$N_6^v(x) = a\cdot cosh\,(La)\cdot sin\,(Lb)\cdot sinh\,(ax)\cdot sin\,(bx) - a\cdot sinh(La)\cdot sin(Lb)\cdot cosh(ax)\cdot sin(bx) - b\cdot cos(Lb)\cdot sinh(La)\cdot sinh(ax)\cdot sin(bx) + \frac{b\cdot sinh(La)\cdot sin(Lb)\cdot cos(bx)\cdot sinh(ax)}{-a^2cos(Lb)^2 + a^2 - b^2cosh(La)^2 + b^2} \quad (13)$$

$$N_6^\theta(x) = -(b^2\cdot cos\,(Lb)\cdot sinh\,(La)\cdot cos\,(bx)\cdot sinh\,(ax) - a^2\cdot cosh(La)\cdot sin(Lb)\cdot cosh(ax)\cdot sin(bx) + a^2\cdot sinh(La)\cdot sin(Lb)\cdot sinh(ax)\cdot sin(bx) + b^2\cdot sinh(La)\cdot sin(Lb)\cdot sinh(ax)\cdot sin(bx) - a\cdot b\cdot cosh(La)\cdot sin(Lb)\cdot cos(bx)\cdot sinh(ax) + \frac{a\cdot b\cdot cos(Lb)\cdot sinh(La)\cdot cosh(ax)\cdot sin(bx))}{-a^2cos(Lb)^2 + a^2 - b^2\cdot cosh(La)^2 + b^2} \quad (14)$$

The axial interpolation functions are given by eq. (15), obtained from conventional linear interpolations.

$$N_1^v(x) = 1 - \frac{x}{L} \qquad N_4^v(x) = 1 - \frac{x}{L} \quad (15)$$

### 2.3 Stiffness Matrix

By using the Updated Lagrangian formulation, proposed by McGuire et al. [7], the equation for developing the element's elastic stiffness matrix $[K_{re}]$ is obtained according eq. (16). The stiffness matrix consists of two terms: the first one represents the axial behavior, which is not relevant for the proposed study, and the second one represents the flexural behavior. In this context, $EA$ represents the axial stiffness, $EI_z$ the bending stiffness of the bar, $[N_u]$ the matrix of axial shape functions, and $[N_v]$ the matrix of bending interpolation functions.

$$K_{re} = \{\delta u\}^T \int_0^L EA \cdot [N_u{}'] \cdot [N_u{}']^T dx\{u\} + \{\delta v\}^T \int_0^L EI_z \cdot [N_v{}''] \cdot [N_v{}'']^T dx\{v\} \tag{16}$$

Additionally, since the element is supported by springs, the effects of the elastic foundation must be considered as adding the elastic stiffness matrix $[K_m]$ shown in eq. (17). Thus, the complete elastic stiffness matrix $[K_g]$ governing the global behavior of the element is given by the sum of these individual terms, as indicated by eq. (18).

$$K_m = \{\delta v\}^T \int_0^L k \cdot [N_v] \cdot [N_v]^T dx\{v\} \tag{17}$$

$$K_g = K_{re} + K_m \tag{18}$$

Through this formulation and integration technique, it is possible to determine the behavior of the structural element on an elastic foundation at any point without the need for discretization.

## 3 Numerical Application

To demonstrate the formulations mentioned here, from now on referred to as the "Complete Formulation" or just "Complete", the linear displacement of the concrete structure shown in Fig. 3, will be calculated and compared to the results presented in Teodoru [8] and the ones from usual FEM solution. The properties used were: modulus of elasticity $E = 21700MPa$, soil spring $k^v = 22MN/m^3$, height of 0,6m and width of 2,64m.

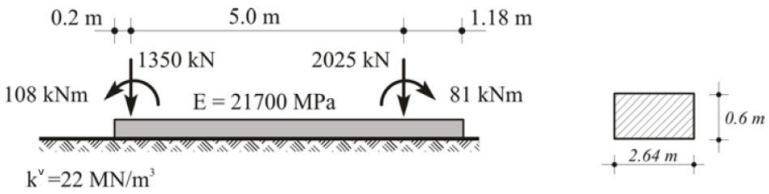



Figure 3. RC beam over elastic foundation (adapted from Teodoru [8])

As the smallest possible discretization, three elements were used for the Complete Formulation, since a node is required at each end and also at the points of load application, resulting in the four nodes of Fig. 4, from A to D. For the model presented by Teodoru [8] and the one using FEM, internal nodes were defined as shown in Fig. 5, resulting in 13 nodes. For further comparison, a more refined discretization model was set for the FEM analysis as well, resulting in 65 nodes (Fig. 6). Thus, the results obtained are presented in Tab. 1, showing the vertical displacement and rotation for each of these four main nodes (A to D). The loads are applied at nodes B and C, while A and D corresponds to the final nodes.




Figure 4. Discretization for the Complete Formulation




Figure 5. Discretization for the original analytical and FEM models

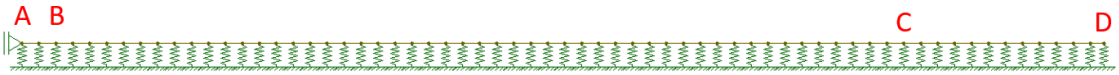



Figure 6. Refined discretization for the FEM model

Table 1. Results comparing the models of Teodoru, FEM and Complete

| MODEL | NUMBER OF NODES | ROTATION ON MAIN NODES (rad) | | | | VERTICAL DISPLACEMENT ON MAIN NODES (m) | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | A | B | C | D | A | B | C | D |
| Teodoru [8] | 13 | -0,0025 | -0,0025 | 0,0010 | 0,0008 | -0,0118 | -0,0113 | -0,0088 | -0,0097 |
| FEM | 13 | -0,0025 | -0,0025 | 0,0010 | 0,0008 | -0,0118 | -0,0113 | -0,0088 | -0,0097 |
| Complete | 4 | -0,0027 | -0,0027 | 0,0015 | 0,0013 | -0,0124 | -0,0118 | -0,0098 | -0,0114 |
| FEM (refined) | 65 | -0,0026 | -0,0026 | 0,0014 | 0,0013 | -0,0120 | -0,0114 | -0,0096 | -0,0112 |

It is possible to observe that the results from the Teodoru [8] model and FEM are identical, while those from the Complete Formulation show a slight discrepancy when compared to the above. However, when a refined discretization is applied to the FEM model, the results tend to converge toward those presented by the Complete Formulation, once the analytical solution should correspond to the Complete Formulation. Figure 7 shows the vertical deflection along the beam for all the models.

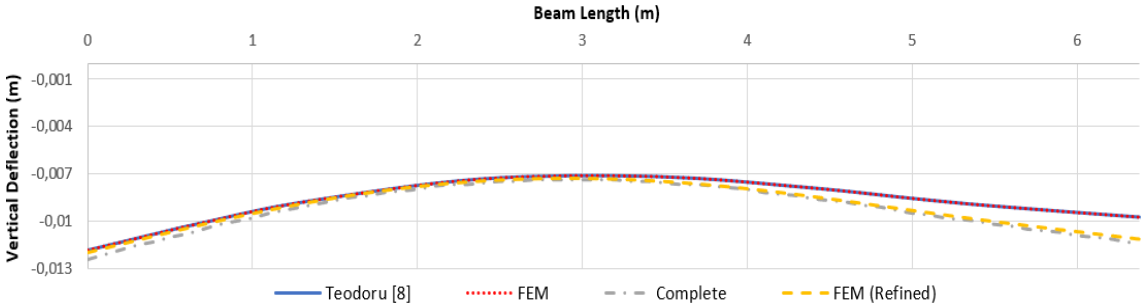



Figure 7. Vertical deflection for all models

Finally, Fig. 8 shows the bending moment diagram for all the models, where the same behavior of before can be seem. It is clear that a refined discretization leads to higher negative bending moments. The reinforcement for this concrete beam should employ a more refined discretization or the Complete Formulation with a reduced mesh.

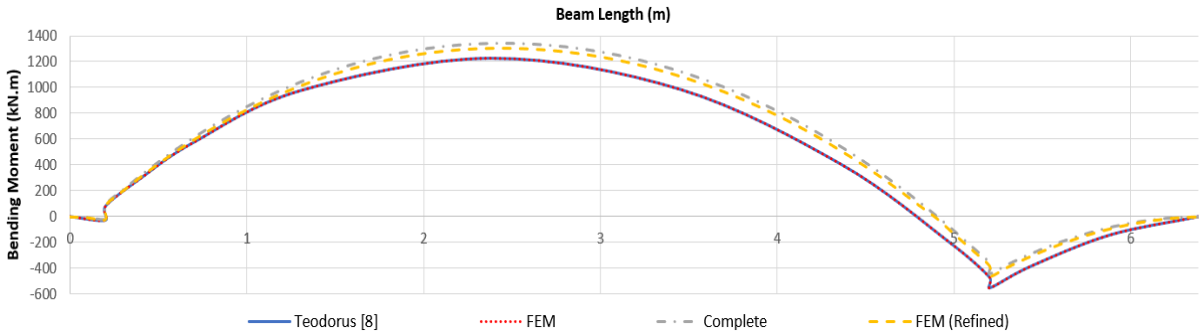



Figure 7. Bending moment for all models

# 4 Conclusions

The analysis of concrete elements on elastic foundations requires the application of techniques such as analytical solutions or the Finite Element Method (FEM). This method, like other numerical approaches, relies on the discretization of the continuous element, which, although efficient, can present certain issues, such as the subjectivity of the discretization, approximate results, computational effort, or the model complexitiy.

Thus, the Complete Formulation proposed the application of equations derived from the problem definition of infinitesimal element on an elastic foundation, together with the shape functions that govern the behavior of the element, to determine transversal displacements, cross-section rotations and bending moment along a continuous beam, with the need off minimum discretization proving to be and simplified and efficient formulation to simulate structures resting on elastic foundations, such as raft foundations and footing beams.

In order to validate the development, the results of the Complete Formulation were compared with those obtained from literature and numerical methods such as FEM. It was observed that the proposed formulation offers greater accuracy to predict the beam behavior, requiring a reduced level of discretization, while in the other methods a refined discretization is necessary to achieve values close to the actual results.

**Acknowledgements.** The authors would like to thank CAPES for the support given to the postgraduate program in civil engineering at UFES.

**Authorship statement.** The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

# References

[1] O. C. Zienkiewicz and R. L. Taylor. The finite element method: its basis and fundamentals. Elsevier, 2005.
[2] S. Liu, J. Wan, C. Zhou, Z. Liu and X. Xang, “Efficient beam–column finite-element method for stability design of slender single pile in soft ground mediums”. International journal of geomechanics, 2019.
[3] L. E. Silva and R. B. Burgos, “Second-order two-cycle analysis of frames based on interpolation functions from the solution of the beam-column differential equation”. REM - International Engineering Journal, v. 76, n. 2, p. 139–146, 2023.
[4] R. B. Burgos and L. F. Martha, “Exact shape functions and tangent stiffness matrix for the buckling of beam-columns considering shear deformation”. In: XXXIV Iberian Latin American Congres on Computational Methods in Engineering, 2013.
[5] M. A. C. Rodrigues. Soluções integradas para as formulações do problema de não linearidade geométrica. PhD thesis, Pontifícia Universidade Católica do Rio de Janeiro, 2019.
[6] L. F. Martha. Análise matricial de estruturas com orientação em objetos. Elsevier, 2018.
[7] W. McGuire, R. H. Gallagher and R. D. Ziemian. Matrix structural analysis. John Wiley & Sons Inc, 2000.
[8] I. B. Teodoru. A computer code for analysing beams on elastic foundation. Intersections, 2009.