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Deformed shape of 2D frames composed of parabolic arc-shaped and straight elements using differential equations

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Abstract. This paper presents the differential relations for displacements, and rotations of parabolic arc-shaped elements subjected to various loads, along with their computational implementation in a plotting algorithm for 2D frames composed of both curved and straight bars. The program is written in Python and is based on the Finite Element Method. The differential equations for straight elements, which are well-established, provide the basis for formulating those applicable to the parabolic arcs. The stiffness matrices are obtained by inverting the flexibility matrices, which are calculated using the Force Method along with a Gauss-Legendre Quadrature applied to evaluate specific integrals. The program is validated against several frame structures analyzed using commercial software, in which the arc segments are approximated by a series of straight elements. Results for displacements and rotations are shown to agree very well with those calculated using the Virtual Work Method, whereas the results from commercial software vary significantly with the level of discretization.

Keywords: Differential relations; Parabolic arc-shaped; Finite Element Method; Python; Gauss-Legendre Quadrature.

1 Introduction

The study of the forces and deformations acting on structures has been conducted since ancient times. Timoshenko [1] highlights that Archimedes already sought to establish principles for the equilibrium of bodies, although it was only from the 17th century, with Galileo Galilei, that structural engineering began to adopt scientific foundations. Nowadays, computational tools play an essential role in modeling and analyzing structures under various loading conditions, especially in curved elements subjected to thermal loads, as discussed by Melo et al. [2].

Among these tools, the PAE (Structural Analysis Program) stands out. Developed in Fortran by the research group at IFMA, it is designed for the analysis of structures with non-conventional characteristics, such as variable moments of inertia and curved elements (parabolic and circular). Previous versions of the PAE have been enhanced with the development of graphical interfaces, as demonstrated by Oliveira et al. [3], and with the implementation of curved elements applied to grillage analysis, as proposed by Lima and Soares [4], in addition to the integration of advanced numerical techniques such as Gaussian quadrature, as implemented by Rego et al. [5]. More recently, Coimbra [6] developed a module focused on the modeling of translational joints in 2D frames, expanding the program's analytical capabilities.

In this context, the present article proposes the creation of a new module for the PAE, focused on determining the elastic curve of bars with parabolic arc geometry through differential relations. It is expected that this work will contribute to improving the visualization and analysis of structural behavior, as well as reinforcing the use of the PAE as a tool to support teaching and research.

2 Methodology

The code was initially conceived for the analysis of structures composed exclusively of straight bars, whose loading vectors and stiffness matrices are well established in the technical literature, as demonstrated by Martha [7]. In the case of elements with parabolic arc geometry, the stiffness matrices and loading vectors were determined by inverting the corresponding flexibility matrices, previously calculated using the Force Method. To ensure accuracy in the integrals involved in this procedure, the Gauss-Legendre numerical method was adopted, known for its high precision in solving definite integrals.

The differential relations applied in this study are consistent with the classical formulations of structural analysis described by Soriano [8], which establish direct relationships between internal forces and displacements in elastic beams.

2.1 Straight Element

The deflected shape was determined using differential equations derived from classical beam theory for straight bars, where internal forces and displacements are directly related through derivatives. According to Martha [7], Navier's 19th-century bending formulation neglects shear deformations, resulting in a differential equation for transverse displacements under distributed loads, valid for beams with constant inertia and subject to the following simplifying assumptions:

- The material behaves in a linear-elastic manner, according to Hooke's law;
- Displacements are small enough to preserve the geometric linearity of the model;
- Cross-sections remain plane and perpendicular to the beam axis after deformation (Bernoulli's hypothesis).

Under these assumptions, the transverse behavior of straight beams can be described by a fourth-order differential equation. This equation, which relates the distributed load to the beam's transverse displacement, is expressed through the sequence of relationships shown in eq. (1), eq. (2), eq. (3), eq. (4), and eq. (5).

$$E \cdot I \cdot v^{IV}(x) = q(x) \tag{1}$$
$$E \cdot I \cdot v^{III}(x) = C(x) \tag{2}$$
$$E \cdot I \cdot v^{II}(x) = M(x) \tag{3}$$
$$v^{I}(x) = \theta(x) \tag{4}$$
$$v(x) = transverse\ displacement \tag{5}$$

In this formulation, E and I represent the Young's modulus of the material and the moment of inertia of the cross-section, respectively, which together define the beam's flexural rigidity. Based on this, the distributed load $q(x)$, shear force $C(x)$, bending moment $M(x)$, rotation $\theta(x)$, and transverse displacement $v(x)$ can be related through successive integrations of the bending differential equation.

These relations allow the determination of the beam's displacements and rotations from the bending moment equation, provided that the appropriate boundary conditions are applied to define the integration constants.

2.2 Parabolic arc element

The analysis of curved beams with a parabolic axis follows the same fundamental principles as for straight elements, but with adaptations to account for curvature. To represent the geometry, the general parabolic equation $y(x) = Ax^2 + Bx + C$ was adopted, fully defining the element's shape. This formulation allows the calculation of nodal displacements (horizontal u, vertical v, and rotational θ) along the structure, as illustrated in the finite element shown in Fig. 1.

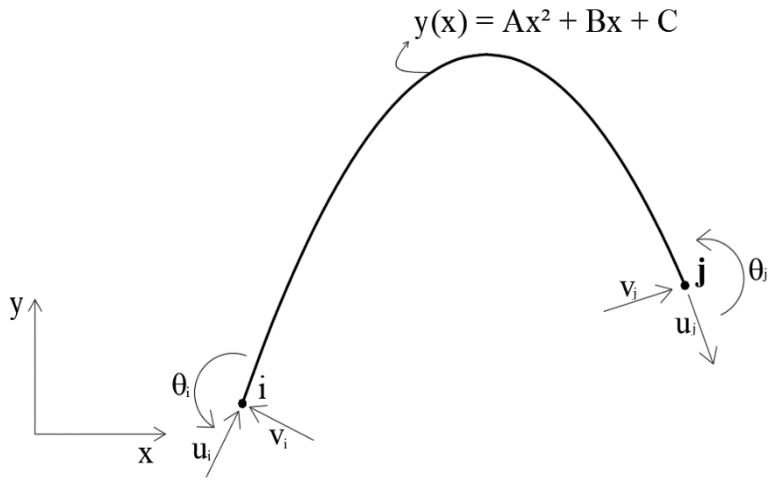


Figure 1: Generic parabolic arc element.

In this formulation, the parameters A, B, and C uniquely define the geometry of the parabolic arc, where A is the quadratic term controlling curvature, B is the linear coefficient, and C is the vertical offset. The adopted finite element (Fig. 1) considers the degrees of freedom as follows:

- u_i and v_i : horizontal and vertical displacements at the initial node.
- θ_i: rotation at the initial node.
- u_j, v_j, and θ_j: corresponding displacements and rotation at the final node.

Because the element's geometry is defined by a parabola in the xy-plane, the actual length along its axis must be considered to compute displacements accurately. For this purpose, an infinitesimal length ds are defined, representing a small segment along the curve. Its expression, derived from differential geometry, is given in eq. (6).

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx \tag{6}$$

Based on the bending moment distribution $M(x)$ along the parabolic arc, successive integrations were carried out to determine the structural displacements, neglecting shear and axial effects. Owing to the element's curved geometry, the actual differential length ds was used in the formulations, resulting in differential equations for rotation, as well as for the vertical and horizontal displacements of the arc.

The following expressions define the differential equations governing the elastic line along the parabolic arc, considering integrations with respect to both the horizontal coordinate x and the actual length ds. Rotation, vertical displacement, and horizontal displacement are given by eq. (7), eq. (8) and eq. (9), respectively.

$$\frac{d\theta(x)}{dx} = \left(\frac{M(x)}{E \cdot I}\right) \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{7}$$

$$\frac{dv(x)}{dx} = \theta(x) \tag{8}$$

$$\frac{du(x)}{dx} = \left(\theta(x) \cdot \frac{dy}{dx}\right) \tag{9}$$

The integration constants were determined from the boundary conditions defined by the arc supports. This allowed obtaining the final expressions for the displacements and plotting the structure's elastic curve.

2.3 General equation

At the end of the process, the displacement equations along the parabolic arc were obtained, considering the following parameters: concentrated loads at the initial node — vertical load *(P),* horizontal load *(F),* and applied moment *(M)*; distributed loads — horizontal (q_x) and vertical (q_y); material properties — modulus of elasticity *(E)* and moment of inertia *(I)*; geometric parameters from the parabolic equation (A, B, and C); and the initial displacements (v_i, u_i, θ_i) previously computed by the program.

3 Results

Figure 2 shows the structural model analyzed, composed of a curved element with a parabolic axis and a straight element, subjected to various loading types, including concentrated forces and distributed loads. For the straight element, the input data are the coordinates of its nodes, while the parabolic element is defined directly from its mathematical equation, which determines its geometry.

Table 1 lists the material and geometric properties used: Young's modulus *(E)*, related to the material, and the moment of inertia *(I)*, associated with the shape and dimensions of the cross-section. Table 2 compares the results obtained with the proposed method, based on differential equations solved in PAE, with those from the FTOOL software. In FTOOL, the parabolic arc was modeled using 40 straight bar elements, a number chosen as a practical balance between computational effort and geometric approximation. Increasing the number of elements improved accuracy but also made the modeling process more time-consuming, and tests with over 70 divisions led to numerical instabilities in the software.

Although the numerical differences between PAE and FTOOL are small, preliminary analyses based on the Virtual Work Method (VWM) suggest that the results obtained with the PAE formulation are closer to analytical expectations. However, due to space limitations, these additional results are not presented here. The main advantage of PAE lies in its ability to represent curved elements directly, simplifying the modeling process and providing continuous results along the entire element, including intermediate points between nodes.

Table 1: Cross-section material and geometric properties.

Material properties	Geometric properties
$E = 20\ GPa$	$I = 0.001\ m^4$

$$y(x) = -0.24 \cdot x^2 + 2.8 \cdot x$$

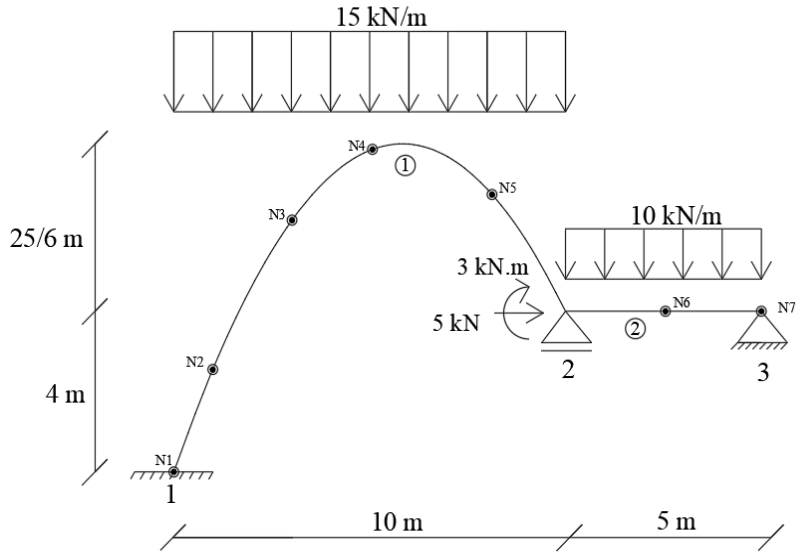


Figure 2: Example used for validation

Table 2: Comparison of results

Node	Method	x (m)	y (m)	Displacement Dx (m)	Error (%)	Displacement Dy (m)	Error (%)	Displacement Rotation (rad)	Error (%)
1	General eq.	0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
	FTOOL			0.0		0.0		0.0	
2	General eq.	1	2.56	$5.2 \cdot 10^{-4}$	5.83	$-2.1 \cdot 10^{-4}$	6.62	$-3.3 \cdot 10^{-4}$	5.39
	FTOOL			$4.897 \cdot 10^{-4}$		$-1.961 \cdot 10^{-4}$		$-3.122 \cdot 10^{-4}$	
3	General eq.	3	6.24	$1.5 \cdot 10^{-3}$	6.53	$-7.72 \cdot 10^{-4}$	7.25	$-6 \cdot 10^{-5}$	6.73
	FTOOL			$1.402 \cdot 10^{-3}$		$-6.676 \cdot 10^{-4}$		$-5.596 \cdot 10^{-5}$	
4	General eq.	5	8	$1.27 \cdot 10^{-3}$	6.53	$-3.8 \cdot 10^{-4}$	7.83	$3.6 \cdot 10^{-4}$	5.75
	FTOOL			$1.187 \cdot 10^{-3}$		$-3.502 \cdot 10^{-4}$		$3.393 \cdot 10^{-4}$	
5	General eq.	8	7.04	$1.66 \cdot 10^{-3}$	6.86	$9 \cdot 10^{-4}$	6.54	$2.5 \cdot 10^{-4}$	4.6
	FTOOL			$1.546 \cdot 10^{-3}$		$8.411 \cdot 10^{-4}$		$2.385 \cdot 10^{-4}$	
6	General eq.	12.5	4	0.0	0.0	$-3.12 \cdot 10^{-3}$	6.22	$-1.13 \cdot 10^{-4}$	7.7
	FTOOL			0.0		$-2.926 \cdot 10^{-3}$		$-1.12 \cdot 10^{-4}$	
7	General eq.	15	4	0.0	0.0	0.0	0.0	$2.1 \cdot 10^{-3}$	6.3
	FTOOL			0.0		0.0		$-1.968 \cdot 10^{-3}$	

4 Conclusions

This study developed and validated a computational method, based on differential equations, for analyzing displacements in 2D frames composed of straight elements and parabolic arcs. Implemented in Python using the Finite Element Method, the algorithm combines flexibility matrix inversion (Force Method) with Gauss–Legendre quadrature to solve differential equations adapted to curved geometries. Validation through comparison with commercial software that models arcs as straight-element discretizations indicated consistent agreement and reasonable accuracy, showing that the proposed formulation provides reliable results even with moderate discretization levels. The differential relations formulated for parabolic arcs, accounting for the actual length ds and integrations along the curved axis, extend Navier's theory to nonlinear geometries. These results confirm PAE as a robust and reliable tool for teaching and research in structural engineering.

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