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# Advanced Creep Modeling of Acetal Using Variable-Order Fractional Mechanics

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**Abstract.** Creep is a time-dependent deformation that occurs when a material is subjected to a constant load, and it plays a critical role in the long-term performance of polymers, potentially leading to rupture or static fatigue. Accurate predictions of polymer behavior under creep conditions are primarily derived from experimental data obtained under controlled environmental settings. Traditional mathematical models, such as the Kelvin–Voigt and Maxwell models—based on combinations of springs and dashpots—exhibit limited accuracy, as they inherently produce exponential responses, whereas experimental creep curves in polymers are better represented by power-law behavior. The springpot, a fractional-order element that bridges the behavior of springs and dashpots, captures this power-law response. However, its fixed-order formulation fails to reflect the time-dependent evolution of the creep exponent observed in real materials. To address this limitation, this study investigates the performance of a recently proposed variable-order springpot model in representing the creep behavior of Acetal polymer.

**Keywords:** nonlinear mechanics, time-dependent deformation, creep of polymers, variable-order fractional calculus, rheological model

## 1 Introduction

Acetal, also known as polyoxymethylene (POM), is a high-performance engineering thermoplastic widely used in precision components due to its excellent dimensional stability, low friction, high stiffness, and resistance to wear and moisture. These properties make Acetal an ideal candidate for applications in automotive, consumer electronics, and mechanical systems, where long-term reliability is essential. However, like most polymers, Acetal exhibits time-dependent mechanical behavior when subjected to constant stress, a phenomenon known as creep. Under prolonged loading, the material undergoes progressive deformation, which may compromise dimensional accuracy or even lead to mechanical failure over time. Understanding and predicting the creep response of Acetal is thus critical for ensuring its structural integrity in service. Although conventional viscoelastic models provide some insight into creep behavior, they often fail to capture the complex, nonlinear, and time-evolving characteristics observed in experimental data. This motivates the development and application of more advanced modeling approaches that can represent the real mechanical response of Acetal under long-term loading conditions.

To model the creep behavior of polymers such as Acetal, rheological representations based on combinations of springs and dashpots have been traditionally employed. Classical models like Maxwell, Kelvin–Voigt, and their combinations shown in Fig. 1 are widely used for their simplicity and ability to capture fundamental viscoelastic responses. However, these models inherently produce exponential-type solutions, which fail to accurately describe the power-law behavior observed in polymer creep experiments, as first reported by Nutting [1]. As a result, fitting experimental data using classical models often requires multiple elements in series and/or parallel, leading to overly complex structures with limited physical interpretability

In response to these limitations, fractional-order models have emerged as a more suitable alternative, with the springpot element playing a central role. One of the earliest applications of fractional calculus in viscoelas-

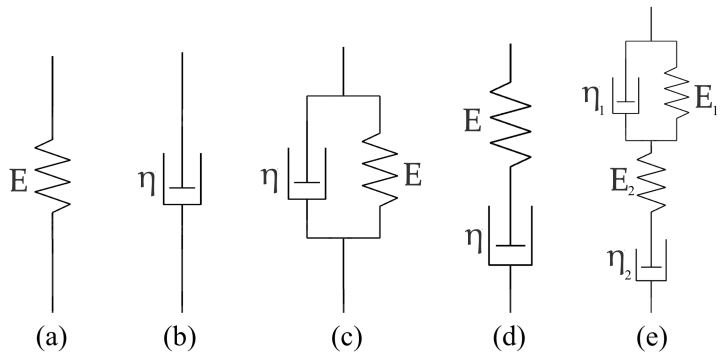



Figure 1. Viscoelastic models: (a) spring; (b) dashpot; (c) Kelvin-Voigt; (d) Maxwell; (e) Burger

ticity was introduced by Gemant [2] in the 1930s during his investigation of mechanical relaxation phenomena in complex fluids. Through experimental observations, Gemant identified that the stress relaxation behavior of certain materials deviated significantly from the exponential decay predicted by classical models such as Maxwell's. Instead, the stress was found to decay according to a power-law indicating the need for a more general mathematical framework. To model this behavior, Gemant [2] proposed the use of fractional derivatives in the constitutive equations, marking one of the first instances of fractional calculus applied to viscoelastic materials. Although he acknowledged the mathematical usefulness of fractional operators in describing such behavior, he expressed skepticism about their physical meaning, considering them primarily empirical tools.

Building on Gemant's pioneering work, Blair [3] further advanced the application of fractional derivatives in modeling time-dependent responses of viscoelastic materials. In an effort to provide a physically interpretable framework capable of representing behavior that lies between purely elastic and purely viscous responses, Blair introduced the concept of the *springpot*—a fractional-order element that generalizes the classical spring and dashpot. This element became a cornerstone of fractional viscoelasticity, enabling the development of compact yet highly accurate rheological models that effectively capture the experimentally observed power-law behavior in both creep and relaxation phenomena. The applicability of fractional calculus in modeling the viscoelastic behavior of polymers was further supported by Di Paola et al. [4], who experimentally verified its validity and demonstrated its superiority over classical integer-order models in representing the time-dependent mechanical response, particularly in regimes where power-law behavior is observed.

The traditional springpot also presents an important limitation: it models power-law behavior using a *constant exponent*, while experimental data for polymers such as Acetal often show that the creep exponent evolves over time. To address this, Bonfanti et al. [5] have demonstrated that the springpot can be used not only as a replacement for dashpots in classical rheological models but also in combination with other springpots, arranged in series or parallel, to construct fully fractional viscoelastic models, as illustrated in Fig. 2. These configurations offer more compact yet highly flexible frameworks capable of capturing a broad spectrum of time-dependent behaviors observed in polymers.

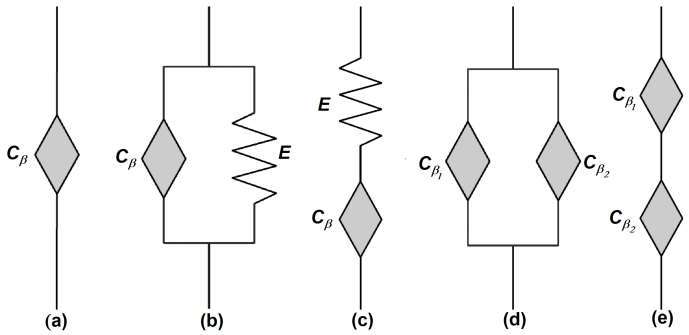



Figure 2. Fractional models: (a) springpot; (b) Fractional Kelvin-Voigt; (c) Fractional Maxwell; (d) springpots in parallel; (e) springpots in series

However, the assumption of a fixed fractional order still limits the accuracy of these models when the material

response evolves dynamically. This limitation has motivated the development of *variable-order fractional models*, as proposed by Di Paola et al. [6], in which the order of the fractional derivative is allowed to vary throughout the deformation process. These models are designed to more accurately reflect the *nonlinear and evolving viscoelastic behavior* of polymers.Within this context, the present study investigates the performance of a *variable-order springpot formulation*, recently proposed by Telles Ribeiro and Cunha [7], in modeling the creep behavior of Acetal. The formulation is evaluated using experimental data to assess its accuracy and practical applicability.

## 2 Fundamentals of Fractional Calculus for Rheological Modeling

As highlighted by Failla and Zingales [8], fractional calculus is a well-established and versatile tool in engineering sciences, particularly effective for modeling the complex behaviors of materials with memory effects. It generalizes classical calculus by allowing derivatives and integrals of non-integer (fractional) order, providing a robust mathematical framework for describing the memory-dependent and hereditary phenomena commonly observed in viscoelastic materials. Among the various formulations, the Riemann–Liouville fractional integral is one of the most widely employed. This formulation extends Cauchy's formula for repeated integration to arbitrary real orders, enabling the modeling of processes that exhibit long-term memory while maintaining a balance between mathematical generality and computational feasibility.

The classical Cauchy formula for repeated integration is expressed as:

$$\int_0^t \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_1} f(\tau)\, d\tau\, d\tau_1 \cdots d\tau_{n-1} = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1}$$

which expresses an $n$-fold integral as a single integral involving a power-law kernel. By extending the integer order $n \in \mathbb{Z}$ to a real order $\alpha \in \mathbb{R}^+$, the *Riemann–Liouville fractional integral* of order $\alpha$ is obtained:

$${}_0\mathcal{I}_t^{\,\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \quad \text{for } t > 0, \tag{2}$$

where $\Gamma(\alpha)$ is the gamma function, a continuous extension of the factorial function to real and complex arguments. This integral operator introduces a *memory effect* through its nonlocal kernel, allowing the current state of the system to depend on its entire deformation history. As such, it is particularly relevant in modeling the *hereditary behavior* observed in viscoelastic materials, especially in creep and relaxation processes.

A more generalized formulation, introduced by Colombaro et al. [9], defines fractional integration with respect to another function $g(t)$, rather than the identity function. This extension enables the fractional operator to incorporate non-uniform weighting of past states, which is particularly useful in describing systems where the memory effect evolves nonlinearly with time. The fractional integral of $f(t)$ with respect to $g(t)$ is given by:

$${}^{g}\mathcal{I}^{\,\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g'(\tau)\ (g(t) - g(\tau))^{\alpha-1}\ f(\tau)\, d\tau \quad \text{for} \quad t > 0\,. \tag{3}$$

This generalized form enhances the flexibility of fractional models, allowing them to better capture the time-dependent characteristics of viscoelastic materials whose behavior cannot be fully described by constant-order fractional operators.

## 3 Fractional Calculus in Viscoelastic Modeling

### 3.1 Constant order springpot

The Boltzmann principle states that the total strain (stress) in a viscoelastic material due to a time-varying stress (strain) is the sum of the strain (stress) caused by each individual increment applied over time. For strain, it can be mathematically written as:

$$\varepsilon(t) = \int_0^t J(t-\tau)\, \frac{d\sigma(\tau)}{d\tau}\, d\tau, \tag{4}$$

where $J(t)$ is the creep function, described as:

$$J(t) = \frac{1}{C_\beta \Gamma(\beta+1)}\, t^\beta \qquad 0 < \beta < 1. \tag{5}$$

By substituting Eq.5 into Eq.4, the fractional constitutive stress-strain relationship is derived as:

$$\varepsilon(t) = \frac{1}{C_\beta \Gamma(\beta+1)} \int_0^t (t-\tau)^\beta \, \frac{d\sigma(\tau)}{d\tau} \, d\tau = \frac{1}{C_\beta} \, \mathcal{I}^{\beta+1} \frac{d\sigma}{dt}(t) = \frac{1}{C_\beta} \, \mathcal{I}^{\beta} \sigma(t) \qquad 0 < \beta < 1. \tag{6}$$

The parameters $C_\beta$ and $\beta$ are intrinsic properties of the material that characterize its viscoelastic behavior. The parameter $C_\beta$, which has units of Pa $\cdot$ s$^\beta$, is often referred to as the *firmness* of the material, as emphasized by Bonfanti et al. [5]. It quantifies the combined influence of the material's dissipative and elastic contributions.

The parameter $\beta$, known as the *order* of the springpot, defines the relative predominance of these two mechanisms. When $\beta = 0$, the element behaves as a purely elastic solid, with $C_\beta$ equivalent to the elastic modulus $E$, i.e., $C_{\beta=0} = E$. Conversely, when $\beta = 1$, the element exhibits purely viscous behavior, and $C_\beta$ corresponds to the dynamic viscosity $\eta$, i.e., $C_{\beta=1} = \eta$.

The relationship between these parameters is given by:

$$C_\beta = \eta^\beta E^{1-\beta} = T^\beta E, \tag{7}$$

where $T = \eta/E$ is the *material timescale*, representing the characteristic time at which viscous effects become comparable to elastic effects.

### 3.2 Variable-order springpot

Experimental results indicate that the rheological model must account for the variation of the parameter $\beta$ over time. Since $C_\beta = T^\beta E$, any change in $\beta$ implies a corresponding variation in $C_\beta$, even if the material timescale $T$ remains constant under a constant load $\sigma_0$. These considerations lead to the formulation of the *variable-order springpot*, whose constitutive equation can be derived from Eq. 3 by adopting $g(t) = t/T$, yielding:

$$\varepsilon(t) = \frac{1}{E\,(\sigma_0)\;\Gamma[\beta(t)]} \int_0^t \frac{1}{T\,(\sigma_0)} \left[\frac{t}{T\,(\sigma_0)} - \frac{\tau}{T\,(\sigma_0)}\right]^{\beta(t)-1} \sigma_0 \, d\tau \,, \tag{8}$$

where $\beta(t)$ denotes the *variable fractional order*. Accordingly, the corresponding variable-order relation is expressed as:

$$\varepsilon(t) = \frac{1}{E\,(\sigma_0)} \, {}^g\mathcal{I}^{\beta(t)} \sigma_0 \,. \tag{9}$$

Since the parameters $\sigma_0$, $\beta(t)$ and $T\,(\sigma_0)$ in Eq.(8) are not a function of $\tau$, and considering the identity $\Gamma[\beta(t)]\,\beta(t) = \Gamma[\beta(t)+1]$, we have

$$\varepsilon(t) = \frac{\sigma_0}{E\,(\sigma_0)\;\Gamma[\beta(t)]} \, \frac{1}{T\,(\sigma_0)^{\beta(t)}} \int_0^t (t-\tau)^{\beta(t)-1} \, d\tau \,, \tag{10}$$

and the the creep response for a *variable-order springpot* given by

$$\varepsilon(t) = \frac{\sigma_0}{E\,(\sigma_0)\;\Gamma[\beta(t)+1]} \left[\frac{t}{T\,(\sigma_0)}\right]^{\beta(t)} \,, \quad 0 \le \beta(t) \le 1 \,. \tag{11}$$

The temporal evolution of the fractional order $\beta(t)$ is modeled as a transition from an initial value $\beta_0$ to a final asymptotic value $\beta_\infty$, satisfying $0 \le \beta_0 < \beta_\infty \le 1$. In the context of creep, the following function is adopted to describe this transition:

$$\beta(t) = \frac{\beta_\infty \left(\frac{t}{\gamma}\right)^\delta + \beta_0}{\left(\frac{t}{\gamma}\right)^\delta + 1} \,, \tag{12}$$

where $\gamma > 0$ is the *characteristic time*, governing the timescale of the transition, and $\delta > 0$ is the *shaping exponent*, which controls the sharpness of the transition from $\beta_0$ to $\beta_\infty$.

## 4 Modelling and results

Note the variable-order springpot model involves six parameters $(E, \eta, \beta_0, \beta_\infty, \gamma, \delta)$, that need to be determined using experimental data. The temporal dependence of $\beta$, encapsulated in Eq.(12), introduces an extra level of complexity into the fractional model.

To address this, it has been employed an optimization approach using the MATLAB package **CEopt** (`https://ceopt.org`), which leverages the Cross-Entropy (CE) method explored by Cunha Jr et al. [10]. The CE method is particularly suitable for this application as it transforms the non-convex optimization problem into a rare event estimation problem, solvable with Monte Carlo simulations [11, 12]. Unlike gradient descent, which may get trapped in local minima, or genetic algorithms, which can be computationally demanding, CE is robust in handling non-convex landscapes, particularly in mechanics applications, presented in the works of Cunha Jr [13], Issa et al. [14], Dantas et al. [15, 16]

The CE method is chosen here for its robustness in exploring parameter spaces and avoiding premature convergence on local minima. It iteratively refines its search, focusing on high-performance areas within the parameter space, which is essential when working with complex, non-linear models like the variable-order springpot. This process allows CE to achieve a stable, physically consistent calibration that is less dependent on initial parameter guesses.

The proposed misfit function for the model calibration problem is defined as:

$$\mathcal{J}(E,\eta,\beta_0,\beta_\infty,\gamma,\delta) = \sqrt{\frac{\displaystyle\sum_{i=1}^{N}\left(\varepsilon^i_{exp} - \varepsilon^i(E,\eta,\beta_0,\beta_\infty,\gamma,\delta)\right)^2}{\displaystyle\sum_{i=1}^{N}\left(\varepsilon^i_{exp}\right)^2}} \times 100\,, \tag{13}$$

where $N$ is the number of points sampled from the curves, $\varepsilon^i_{exp}$ are the values sampled from experimental curve at instants $t_i$, $\varepsilon^i$ are the predicted values from Eq.(11). The misfit value $\mathcal{J}$ is a measure of the percentage deviation between theoretical and experimental curves.

The viscosity parameter $\eta$ represents the microscopic chain-to-chain friction within the polymer and is expected to remain essentially independent of stress. Modeling results confirmed that the best and most consistent fit was obtained by keeping this parameter constant at $\eta = 1000$ GPa·s. Other parameters that are best represented as constants are $\delta = 0.6$ and $\beta_0 = 0.0469$. The remaining parameters exhibit load dependence, with their corresponding values listed in Tab. 1, and their variation with stress expressed as:

$$\begin{cases} E = -7.25 \cdot \sigma_0 + 2567.38 \ \text{ MPa} \\ \beta_\infty = 0.0003 \cdot \sigma_0 + 0.0864 \\ \gamma = \dfrac{11031.18 \cdot (\sigma_0/2.04)^{4.66} + 3288.04\,\sigma_0 + 326714}{(\sigma_0/2.04)^{4.66} + 1} \ \text{ s}, \end{cases} \tag{14}$$

Table 1. Estimated parameter values for modeling the creep behavior of Acetal at 20°C under various applied stresses ($\sigma_0$). The parameters $E$ (elastic modulus), $\eta$ (viscosity), $\beta_0$ (initial fractional order), $\beta_\infty$ (asymptotic fractional order), $\gamma$ (transition rate) and $\delta$ (shaping exponent) were calibrated for each stress level. Misfit values indicate the accuracy of model fit to the experimental data, demonstrating good adherence across stress conditions.

| $\sigma_0$ (MPa) | $E$ (MPa) | $\eta$ (GPa . s) | $\beta_0$ | $\beta_\infty$ | $\gamma$ (s) | $\delta$ | Misfit (%) |
|---|---|---|---|---|---|---|---|
| 2.5 | 2.51e+03 | 1000 | 0.0470 | 0.0869 | 1.01e+05 | 0.60 | 0.15 |
| 5.0 | 2.57e+03 | 1000 | 0.0469 | 0.0886 | 1.37e+04 | 0.60 | 0.68 |
| 7.5 | 2.51e+03 | 1000 | 0.0503 | 0.0884 | 1.80e+04 | 0.60 | 0.50 |
| 10.0 | 2.50e+03 | 1000 | 0.0484 | 0.0900 | 1.26e+04 | 0.60 | 0.82 |
| 12.5 | 2.47e+03 | 1000 | 0.0449 | 0.0901 | 9.31e+03 | 0.60 | 0.94 |
| 15.0 | 2.45e+03 | 1000 | 0.0437 | 0.0914 | 7.33e+03 | 0.60 | 0.97 |

Together, Eqs. (11), (12), and (14) establish a system that reliably models the creep behavior of Acetal at 20°C for applied loads between 2.5 and 15 MPa. This system combines both variable-order fractional calculus

and phenomenological modeling to capture the complex viscoelastic response of polypropylene. The comparison in Fig. 3 between the model's calculated deformation curves and the experimental data highlights the model's effectiveness, with close alignment over a broad range of applied stresses.

Initially, as shown in Fig. 4 (a), $\beta(t)$ remains stable in the glassy phase, where Acetal exhibits predominantly elastic behavior, resisting deformation with minimal energy dissipation. Similarly, Fig. 4 (b) shows how the firmness parameter $C_\beta$ evolves under varying stress conditions and over time.

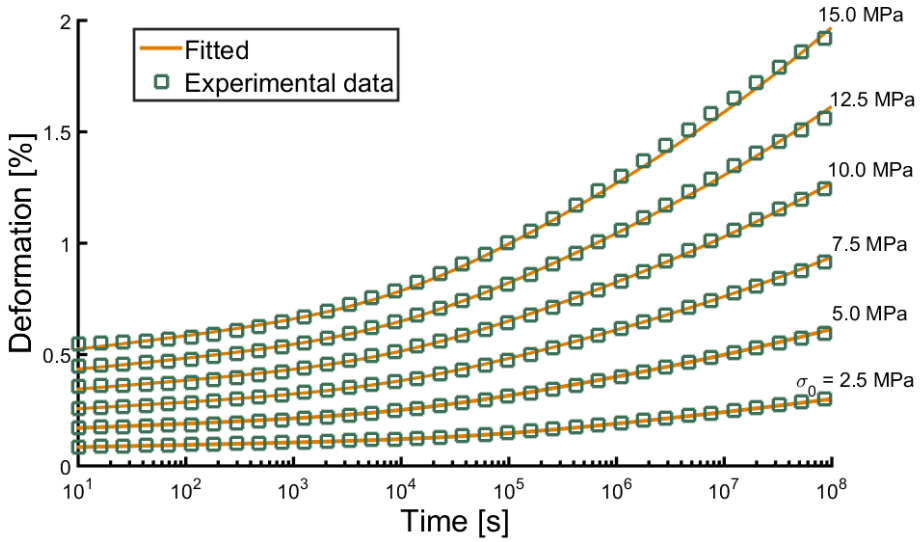



Figure 3. Comparison of experimental and modeled creep curves for Acetal at 20°C under constant stress levels between 2.5 and 15 MPa.

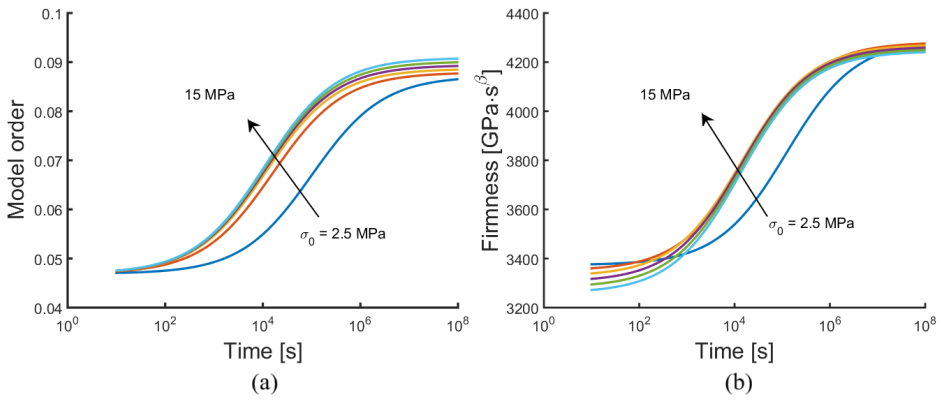



Figure 4. Evolution of the order parameter $\beta$ and firmness $C_\beta$ for Acetal at 20°C under constant stress levels between 2.5 and 15 MPa. The model order $\beta$ exhibits a monotonic increase over time for all stress levels, reflecting a progressive shift from an elastic-dominated response to a more dissipative, viscoelastic behavior.

## 5 Conclusions

This study introduced a novel variable-order fractional model to predict the creep behavior of polymers at room temperature under different applied loads. By incorporating variable-order fractional calculus, the model successfully captures the evolving viscoelastic response of Acetal across multiple stress levels, providing a clear advantage over conventional models that assume constant parameters.

Allowing the fractional derivative order, $\beta$, to vary with time enables the model to represent the material's gradual transition from elastic-dominant to viscous-dominant behavior during creep. The variable-order nature of $\beta(t)$, as defined by Eq. (12), enhances the model's flexibility by accounting for time-dependent microstructural

rearrangements—such as molecular chain realignment—that naturally occur in polymers like Acetal and influence their mechanical properties.

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