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BEM analysis for coupled single and double beam systems

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Abstract. The static and dynamic behavior of elastically connected structures has been extensively studied due to its numerous practical applications in civil, mechanical, and aerospace engineering. In the last decades, many works have focused on continuously connected double-beam systems (DBS), using numerical or analytical approaches. However, these studies often rely on idealized assumptions, such as continuous connections, identical beam lengths, and classical boundary conditions. In this work, a numerical formulation based on the boundary element method (BEM) for the bending analysis of DBS partially connected by elastic layer is presented. The Euler-Bernoulli beam theory is used to represent the beams, while Winkler model is used for the connecting layer. A BEM coupling technique based on sub-domains is employed to model the problem, which combines single-beam and double-beam boundary elements to represent the partial connection. Numerical examples considering arbitrary flexural rigidities, classical and non-classical boundary conditions are presented. The results demonstrate the accuracy and efficiency of the proposed formulation, and highlight its applicability to a wide range of DBS configurations.

Keywords: BEM Coupling problem, Euler-Bernoulli, Partially connected DBS, Winkler

1 Introduction

Elastically connected structures, composed of two or more beams or plates joined by an elastic layer, are widely used in civil, aerospace, and mechanical engineering. Due to their versatility in applications such as sandwich beams, continuous dynamic absorbers, contacting machine components, and even carbon nanotubes, double-beam systems have attracted considerable interest from the scientific community in recent decades.

Double-beam systems have been the focus of many works exploring their dynamic behavior [1–5], the layer's effect on buckling [6–9], and some on the bending analysis [10, 11]. Predominantly, the Euler-Bernoulli beam theory is usually used, and analytical or numerical techniques like the finite element method (FEM) are used to conduct the analysis. However, boundary element method (BEM) applications remain scarce, with few contributions such as [11]. Furthermore, most studies rely on idealized models that assume continuous connections, identical beam lengths, and classical boundary conditions, which do not accurately represent practical applications involving partial connections or varying beam lengths.

There are a few studies that do consider such complexities [12–16]. Mirzabeigy and Madoliat [12] studied the dynamic behavior of the DBS partially connected with elastically restrained ends. The natural frequencies and mode shapes were obtained using the differential transform method. Additionally, Lee et al. [13] studied the dynamic behavior through the transfer matrix method. Liu and Yang [14] presented the distributed transfer function method for vibration analysis of DBS with elastically restrained ends. Fei et al. [15] presented the dynamic modeling and modal analysis of a double-beam system with different beam lengths connected by discrete translational springs using the dynamic stiffness method. More recently, based on the discrete technique and the transfer matrix method, Zhou et al. [16] studied the vibration analysis of a partially viscoelastic connected Timoshenko double-beam system with a non-uniform cross-section.

To address these more practical scenarios, this paper presents a direct BEM formulation for the bending analysis of double-beam systems partially connected by a Winkler layer. The coupling technique is based on dividing the structure's domain into sub-domains, analyzing the structural elements as separate single-beam or double-beam elements, and subsequently applying the continuity and equilibrium conditions to couple the system.

2 BEM formulation

As shown in Fig. 1, the double-beam system (DBS) consists of two upper and lower parallel beams connected along their entire domain or only in parts of it by an elastic layer. The beams can be of the same or different lengths. The DBS assumptions are that the kinematic behavior of the beams is based on the classical Euler-Bernoulli beam theory and the elastic layer is based on the Winkler model, which consists of a series of linear and independent springs with rigidity c.

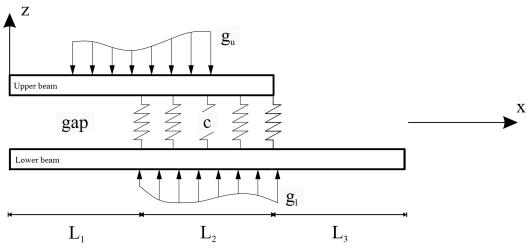


Figure 1. Double-beam system partially connected with beams of different lengths

The BEM formulation is used to model the DBS, as shown in Fig. 1, or any other similar system, as a combination of a single-beam boundary element and a double-beam boundary element. Both are 1D elements with four and eight degrees of freedom, respectively. The coupling strategy is based on analyzing the structural elements disconnectedly and applying the compatibility of displacements and equilibrium conditions at all nodes in contact. All steps to obtain the algebraic equations are shown below.

2.1 Single-beam integral equation

The differential equilibrium equation for the single-beam problem considering the Euler-Bernoulli beam theory is given by

$$EI\frac{d^4w}{dx^4} - g = 0, \tag{1}$$

where $w = w(x)$ is the vertical displacement of the longitudinal axis of the beam, E is the modulus of elasticity, I is the moment of inertia of the cross-section, and $g = g(x)$ is the distributed vertical loading intensity. Also, the bending moment and shear force are defined as:

$$M(x) = -EI\frac{d^2w}{dx^2},\ V(x) = -EI\frac{d^3w}{dx^3}. \tag{2}$$

The required boundary integral equation can be obtained by applying the method of weighted residuals to the beam's equilibrium equation and integrating over the beam domain. In the direct BEM formulation, the weight function is a solution of the governing differential equation for an infinite domain subject to a unit load in terms of Dirac's delta, as can be seen in reference [17]. Thus, after applying the method of weighted residuals and performing some integration by parts, the integral equation that provides the transverse displacement at any interior point is obtained:

$$w(\widehat{x}) + [V^*w]_0^L - [M^*w_{,x}]_0^L = [Vw^*]_0^L - \left[Mw^*_{,x}\right]_0^L + \int_0^L gw^*dx, \tag{3}$$

where the fields with the superscript $*$ are the fundamental solutions and the derivatives are denoted by subscripts preceded by a comma.

It is now is necessary to obtain an additional equation, as two boundary conditions are unknown. This integral equation can be obtained by differentiating Eq.(3) with respect to the source-point $\widehat{x}$ as follows

$$w_{,\widehat{x}}(\widehat{x}) + \left[V^*_{,\widehat{x}} w\right]_0^L - \left[M^*_{,\widehat{x}} w_{,x}\right]_0^L = \left[V w^*_{,\widehat{x}}\right]_0^L - \left[M w^*_{,x\widehat{x}}\right]_0^L + \int_0^L g w^*_{,\widehat{x}} dx, \tag{4}$$

which provides the slopes at any interior point of the beam domain.

2.2 Double-beam integral equation

The equilibrium equations of the DBS with an intermediate Winkler layer, based on Euler-Bernoulli beam theory, are given as follows:

$$E_u I_u \frac{d^4 w_u}{dx^4} + c\,(w_u - w_l) = g_u, \tag{5}$$

$$E_l I_l \frac{d^4 w_l}{dx^4} - c\,(w_u - w_l) = g_l, \tag{6}$$

where $w_i = w_i(x)$ is the beam's displacement, E_i is the modulus of elasticity, I_i is the moment of inertia of the cross-section, c is the Winkler stiffness modulus, and the subscript $i = u, l$ denoting the upper and lower beams, respectively. The bending moment and shear force are defined as:

$$M_i(x) = -E_i I_i \frac{d^2 w_i}{dx^2},\; V_i(x) = -E_i I_i \frac{d^3 w_i}{dx^3}. \tag{7}$$

The solution to the problem can be derived by transforming the differential equilibrium equations into equivalent integral equations. This is achieved by applying the method of weighted residuals to Eqs. (5) and (6). Thus, the integral equations for the transverse displacements at any interior point of the upper or lower beam can be expressed as follows:

$$\begin{aligned} w_i(\widehat{x}) + \left[V^{*i}_u w_u\right]_0^L - \left[M^{*i}_u w_{u,x}\right]_0^L + \left[V^{*i}_l w_l\right]_0^L - \left[M^{*i}_l w_{l,x}\right]_0^L = \left[V_u w^{*i}_u\right]_0^L \\ - \left[M_u w^{*i}_{u,x}\right]_0^L + \left[V_l w^{*i}_l\right]_0^L - \left[M_l w^{*i}_{l,x}\right]_0^L ,, \end{aligned} \tag{8}$$

where the fundamental solutions are presented in reference [11].

Note that two additional integral equations are required. Thus, differentiating Eq.(8) with respect to the souce point $\widehat{x}$ results in the following expressions:

$$\begin{aligned} w_{i,\widehat{x}}(\widehat{x}) + \left[V^{*i}_{u,\widehat{x}} w_u\right]_0^L - \left[M^{*i}_{u,\widehat{x}} w_{u,x}\right]_0^L + \left[V^{*i}_{l,\widehat{x}} w_l\right]_0^L - \left[M^{*i}_{l,\widehat{x}} w_{l,x}\right]_0^L = \left[V_u w^{*i}_{u,\widehat{x}}\right]_0^L \\ - \left[M_u w^{*i}_{u,x\widehat{x}}\right]_0^L + \left[V_l w^{*i}_{l,\widehat{x}}\right]_0^L - \left[M_l w^{*i}_{l,x\widehat{x}}\right]_0^L , \end{aligned} \tag{9}$$

which provides the slopes at any interior point of the upper or lower beam.

2.3 Algebraic equations

The integral equations obtained in the last sections provide the quantities at interior points of the beam's domain and are written in terms of boundary variables and distributed transverse loading intensity. These integral equations can be transformed into an algebraic representation by discretizing the problem, using boundary elements and evaluating the required integrals. Since the governing equations of a single beam and double-beam are defined as a one-dimensional domain, the boundary elements for both are formed as its end points and do not require any interpolation for the boundary variables. To free form member discontinuities, the algebraic representation can be written by collocating the source at the beam ends, i.e., at $\widehat{x} \to 0 = \lim_{\widehat{x}\to 0}(0+\xi)$ and $\widehat{x} \to L = \lim_{\widehat{x}\to L}(L-\xi)$. Thus, the algebraic equations for the boundary points of the single beam can be written as

$$[H]\{U\} = [G]\{P\} + \{F\}, \tag{10}$$

and similarly for a double-beam system as

$$\left[\overline{H}\right]\left\{\overline{U}\right\} = \left[\overline{G}\right]\left\{\overline{P}\right\} + \left\{\overline{F}\right\}, \tag{11}$$

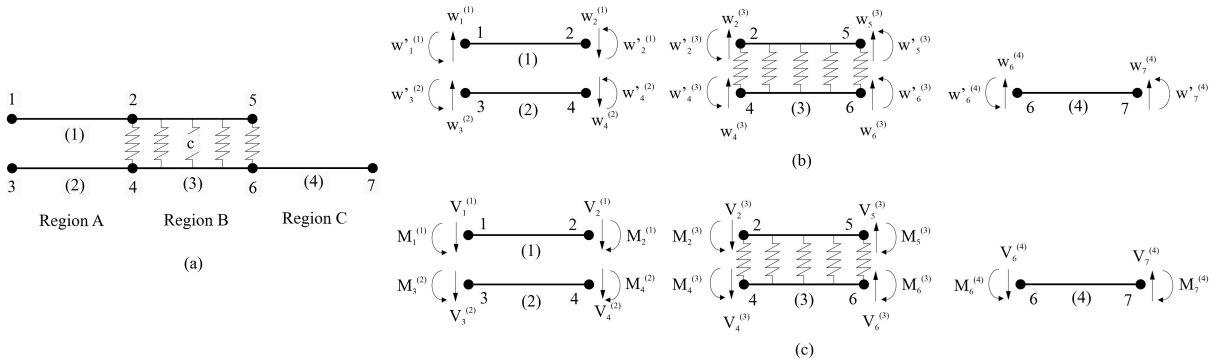


Figure 2. Force and displacement quantities at the region ends

where U and P are the boundary displacements and loads vectors, respectively, H is the influence matrix of displacements, G is the influence matrix of loads, and F is the load vector. The explicit terms of the matrices can be found in the references [11, 17].

In order to present the coupling technique, we consider solving the structure shown in Fig. 1. The strategy adopted in this paper consists of dividing the structure's domain into sub-domains with uniform characteristics, called regions, as shown in Fig. 2. The algebraic representations of each region, whether single-beam or double-beam element, are written using their geometrical and mechanical properties, after which the continuity and equilibrium conditions are set at discontinuity points to find the solution of the structure. Taking into account Fig. 2, the algebraic equations for regions A, B, and C can be written as follows:

$$\begin{aligned}
&Region\ A \begin{cases} [H]^{(1)}\{U_1\} = [G]^{(1)}\{P_1\} + \{F_1\} \\ [H]^{(2)}\{U_2\} = [G]^{(2)}\{P_2\} + \{F_2\} \end{cases}, \\
&Region\ B \begin{cases} \left[\overline{H}\right]^{(3)}\{U_3\} = \left[\overline{G}\right]^{(3)}\{P_3\} + \{F_3\} \end{cases}, \\
&Region\ C \begin{cases} [H]^{(4)}\{U_4\} = [G]^{(4)}\{P_4\} + \{F_4\} \end{cases}, \qquad (12)
\end{aligned}$$

where $\{U_1\} = \left[u_1^{(1)}\ \ u_2^{(1)}\right]^T$, $\{P_1\} = \left[p_1^{(1)}\ \ p_2^{(1)}\right]^T$, $\{U_2\} = \left[u_3^{(2)}\ \ u_4^{(2)}\right]^T$, $\{P_2\} = \left[p_3^{(2)}\ \ p_4^{(2)}\right]^T$, $\{U_3\} = \left[u_2^{(3)}\ \ u_4^{(3)}\ \ u_5^{(3)}\ \ u_6^{(3)}\right]^T$, $\{P_3\} = \left[p_2^{(3)}\ \ p_4^{(3)}\ \ p_5^{(3)}\ \ p_6^{(3)}\right]^T$, $\{U_4\} = \left[u_6^{(4)}\ \ u_7^{(4)}\right]^T$, $\{P_4\} = \left[p_6^{(4)}\ \ p_7^{(4)}\right]^T$, $\{F_1\} = \left[f_1^{(1)}\ \ f_2^{(1)}\right]^T$, $\{F_2\} = \left[f_3^{(2)}\ \ f_4^{(2)}\right]^T$,$\{F_3\} = \left[f_2^{(3)}\ \ f_4^{(3)}\ \ f_5^{(3)}\ \ f_6^{(3)}\right]^T$, and $\{F_4\} = \left[f_6^{(4)}\ \ f_7^{(4)}\right]^T$, where $u_n^{(m)} = \left\{w_n^{(m)}\ \ w_n^{,(m)}\right\}^T$, $p_n^{(m)} = \left\{V_n^{(m)}\ \ M_n^{(m)}\right\}^T$, and the superscript comma denotes the first derivative.

To completely define the algebraic equation of the complete system, after adopting a unified coordinate system, the compatibility of displacements must be satisfied at nodes 2, 4, and 6 by applying the following relations:

$$u_2 = u_2^{(1)} = u_2^{(3)},\ u_4 = u_4^{(2)} = u_4^{(3)},\ u_6 = u_6^{(3)} = u_6^{(4)}. \qquad (13)$$

Furthermore, the equilibrium conditions must also be satisfied:

$$p_2^{(1)} + p_2^{(3)} = \{0\},\ p_4^{(2)} + p_4^{(3)} = \{0\},\ p_6^{(3)} + p_6^{(4)} = \{0\}. \qquad (14)$$

Substituting Eqs. (13) and (14) into Eqs. (12), the algebraic equation for the entire structure can be rewritten as follows:

$$A \cdot X = B \cdot R \qquad (15)$$

where the nonzero matrix coefficients A and B are given as: $A_{1,1} = h_{1,1}^{(1)}, A_{1,2} = h_{1,2}^{(1)}, A_{1,8} = -g_{1,2}^{(1)}, A_{2,1} = h_{2,1}^{(1)}, A_{2,2} = h_{2,2}^{(1)}, A_{2,8} = -g_{2,1}^{(1)}, A_{3,3} = h_{1,1}^{(2)}, A_{3,4} = h_{12}^{(2)}, A_{3,10} = -g_{12}^{(2)}, A_{4,3} = h_{2,1}^{(2)}, A_{4,4} = h_{2,2}^{(2)}, A_{4,10} = -g_{2,2}, A_{5,2} = \overline{h}_{1,1}^{(3)}, A_{5,4} = \overline{h}_{1,2}^{(3)}, A_{5,5} = \overline{h}_{1,3}^{(3)}, A_{5,6} = \overline{h}_{1,4}^{(3)}, A_{5,9} = -\overline{g}_{1,1}^{(3)}, A_{5,11} = -\overline{g}_{1,2}^{(3)}, A_{5,12} = -\overline{g}_{1,4}^{(3)}, A_{6,2} =$

$\overline{h}^{(3)}_{2,1}, A_{6,4} = \overline{h}^{(3)}_{2,2}, A_{6,5} = \overline{h}^{(3)}_{2,3}, A_{6,6} = \overline{h}^{(3)}_{2,4}, A_{6,9} = -\overline{g}^{(3)}_{2,1}, A_{6,11} = -\overline{g}^{(3)}_{2,2}, A_{6,12} = -\overline{g}^{(3)}_{2,4}, A_{7,2} = \overline{h}^{(3)}_{3,1}, A_{7,4} = \overline{h}^{(3)}_{3,2}, A_{7,5} = \overline{h}^{(3)}_{3,3}, A_{7,6} = \overline{h}^{(3)}_{3,4}, A_{7,9} = -\overline{g}^{(3)}_{31}, A_{7,11} = -\overline{g}^{(3)}_{3,2}, A_{7,12} = -\overline{g}^{(3)}_{3,4}, A_{8,2} = \overline{h}^{(3)}_{4,1}, A_{8,4} = \overline{h}^{(3)}_{4,2}, A_{8,5} = \overline{h}^{(3)}_{4,3}, A_{8,6} = \overline{h}^{(3)}_{4,4}, A_{8,9} = -\overline{g}^{(3)}_{4,1}, A_{8,11} = -\overline{g}^{(3)}_{4,2}, A_{8,12} = -\overline{g}^{(3)}_{4,4}, A_{9,6} = h^{(4)}_{1,1}, A_{9,7} = h^{(4)}_{1,2}, A_{9,13} = -g^{(4)}_{1,1}, A_{10,6} = h^{(4)}_{2,1}, A_{10,7} = h^{(4)}_{2,2}, A_{10,13} = -g^{(4)}_{2,1}, A_{11,8} = I, A_{11,9} = I, A_{12,10} = I, A_{12,11} = I, A_{13,12} = I, A_{13,13} = I, B_{1,1} = g^{(1)}_{1,1}, B_{2,1} = g^{(1)}_{2,1}, B_{3,3} = g^{(1)}_{1,1}, B_{4,3} = g^{(2)}_{2,1}, B_{5,7} = \overline{g}^{(3)}_{1,3}, B_{6,7} = \overline{g}^{(3)}_{2,3}, B_{7,7} = \overline{g}^{(3)}_{3,3}, B_{8,7} = \overline{g}^{(3)}_{4,3}, B_{9,10} = g^{(4)}_{1,2},$ *and* $B_{10,10} = g^{(4)}_{22}$. Also, the vectors X, R are defined as

$$X = \left[u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 \;\; u_6 \;\; u_7 \;\; p^{(1)}_2 \;\; p^{(3)}_2 \;\; p^{(2)}_4 \;\; p^{(3)}_4 \;\; p^{(3)}_6 \;\; p^{(4)}_6\right]^T,$$
$$R = [p_1 \;\; 0 \;\; p_3 \;\; 0 \;\; 0 \;\; 0 \;\; p_5 \;\; 0 \;\; 0 \;\; p_7 \;\; 0 \;\; 0 \;\; 0]^T. \tag{16}$$

After applying the boundary conditions and obtaining the boundary displacements and forces, the values of displacements at any interior point of the domain can be obtained using the following algebraic equations for the three regions:

$$\begin{aligned}
&Region\ A \begin{cases} \{U_{d_1}\} + [H_d]^{(1)}\{U_1\} = [G_d]^{(1)}\{P_1\} + \{F_{d_1}\} \\ \{U_{d_2}\} + [H_d]^{(2)}\{U_2\} = [G_d]^{(2)}\{P_2\} + \{F_{d_2}\} \end{cases}, \\
&Region\ B \left\{ \{U_{d_3}\} + \left[\overline{H}_d\right]^{(3)}\{U_3\} = \left[\overline{G}_d\right]^{(3)}\{P_3\} + \{F_{d_3}\} \right. , \\
&Region\ C \left\{ \{U_{d_4}\} + \left[\overline{H}_d\right]^{(4)}\{U_4\} = \left[\overline{G}_d\right]^{(4)}\{P_4\} + \{F_{d_4}\} \right. ,
\end{aligned} \tag{17}$$

where U_{d_i} and P_{d_i} are the displacements and force vectors, respectively, at the interior points of the domains.

3 Numerical example

3.1 Double-beam system partially connected

In this example, the behavior of the structure shown in Fig. 3(a) is studied for different lengths of the elastic layer connecting the upper and lower parts, varying from a fully to a partially connected, and subjected to a uniformly distributed load g_u on the upper beam. The upper and lower beams have different mechanical properties but the same length (L) and cross-section. The parameters were normalized as: $\overline{w}_u = w_u E_u I_u / g_u L^4$, $\overline{w}_l = w_l E_u I_u / \alpha g_u L^4$, $\alpha = E_u I_u / E_l I_l, k = cL^4/E_u I_u$, $\alpha = 2$.

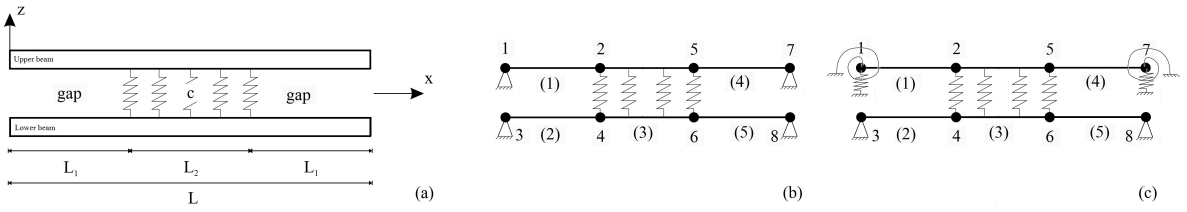


Figure 3. Double-beam system with a partial elastic layer (a) and its corresponding boundary element model with classical (b) and non-classical (c) boundary conditions

The maximal non-dimensional midspan displacements are investigated in two cases: Case 1 considers all ends simply-supported, and Case 2 considers nodes $3, 8$ simply-supported and nodes $1, 7$ with translational and rotational springs with parameters $k_t = 20E_u I_u/L^3$, and $k_r = 0.5E_u I_u/L$, respectively. The results for Case 1 are shown in Table 1 and Figure 4, while those for Case 2 are shown in Table 2 and Figure 5. The structure is modeled as four single-beam boundary elements $(1, 2, 4, 5)$ connected at nodes $2, 4, 5, 6$ to a double-beam boundary element, as shown in Figure 3(b,c).

For Case 1, the maximal non-dimensional displacement on the upper beam decreases as the connection changes from partial ($2L_1/L \to 1$) to complete ($2L_1/L \to 0$). When the ratio tends to be one the upper beam displacement is very close to the classical single beam result $w = 5g_u L^4/384E_u I_u$. On the other hand, for an almost complete connection the values corroborate with the reference [11], as can be seen in Table 1. Case 2 exhibits similar behavior, as can be seen in Table 2.

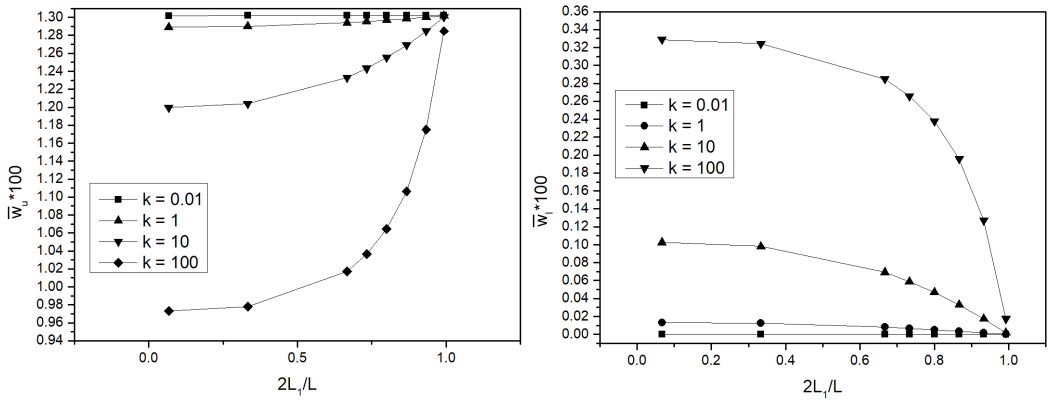


Figure 4. Variation of the maximum non-dimensional midspan displacements with respect to the stiffness and the length of the elastic layer for Case 1

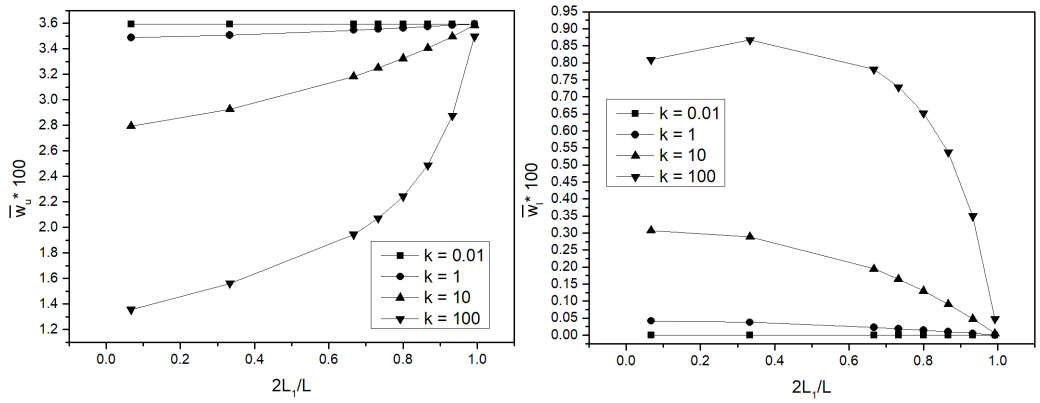


Figure 5. Variation of the maximum non-dimensional midspan displacements with respect to the stiffness and the length of the elastic layer for Case 2

Table 1. Maximal non-dimensional displacements and internal forces for Case 1 (k=100)

Results	Ref. [11]	$2L_1/L$			Results	Ref. [11]	$2L_1/L$		
		0.067	0.667	0.993			0.067	0.667	0.993
$\overline{w}_u \cdot 10^2$	0.9732	0.9733	1.0173	1.2847	$\overline{w}_l \cdot 10^2$	0.3288	0.3288	0.2848	0.0174
$\overline{\phi}_u \cdot 10$	0.3133	0.3133	0.3300	0.4115	$\overline{\phi}_l \cdot 10$	0.2068	0.2067	0.1733	0.0104
$\overline{V}_u$	0.3974	0.3980	0.4281	0.4958	$\overline{V}_l$	0.1026	0.1020	0.0719	0.0042
$\overline{M}_u \cdot 10$	0.9259	0.9260	0.9493	1.2292	$\overline{M}_l \cdot 10$	0.3241	0.3240	0.3007	0.0208

Table 2. Maximal non-dimensional displacements and internal forces for Case 2 (k=100)

Results	Ref. [11]	$2L_1/L$			Results	Ref. [11]	$2L_1/L$		
		0.067	0.667	0.993			0.067	0.667	0.993
$\overline{w}_u \cdot 10^2$	1.3127	1.3563	1.9447	3.4956	$\overline{w}_l \cdot 10^2$	0.7806	0.8087	0.7804	0.0479
$\overline{\phi}_u \cdot 10^2$	1.3193	1.2485	1.4326	3.2184	$\overline{\phi}_l \cdot 10^2$	2.5176	2.6060	2.3759	0.1437
$\overline{V}_u \cdot 10$	1.7472	1.8819	3.0251	4.8851	$\overline{V}_l \cdot 10$	3.2528	3.1181	1.9749	0.1149
$\overline{M}_u \cdot 10$	0.4436	0.4206	0.3569	1.0318	$\overline{M}_l \cdot 10$	0.7404	0.7669	0.8215	0.0573

In both cases, an increase in the stiffness layer modulus results in a reduction of the maximum displacement in the upper beam and a corresponding increase in the lower beam. This effect is more pronounced for high values of k. The transition to a partial connection demonstrates a decrease in the system's stiffness.

4 Conclusions

In this paper, the Boundary Element Method is used for the bending analysis of elastically connected double-beam systems. A gap in the literature is solved by presenting a coupled technique that combines single-beam and double-beam elements for solving DBS partially or completely connected by a Winkler layer and with different beam lengths. A numerical example mixing classical and non-classical boundary conditions is provided in order to validate the coupling technique through a behavior analysis of a double-beam system with connection layer length variation. The present formulation is very attractive due to its versatility and efficiency.

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