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Deflected shape and internal forces of grillage structures using differential equations: PAE-Grelhas 3.0

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Abstract. This paper aims to incorporate differential equations for internal forces, displacements and rotations into the processing of PAE-Grelhas 2.0, a free linear-elastic analysis software for grillage structures composed of straight and circumferential arc-shaped elements. The program, which already includes a graphical user interface and nodal calculations, was initially developed in Fortran, later transcribed into Python, and is based on the Finite Element Method. The main objective is the determination of the deflected shape and internal force diagrams, based on these differential equations, which include bending, torsion and shear force, as well as vertical displacements, and in-plane rotations throughout the domain of the respective elements. Results are compared with those obtained from grillage software where arcs are simulated through many straight elements. Finally, the expressions will be integrated into the program, and the updated version (PAE-Grelhas 3.0) will be made available to the scientific community.

Keywords: Differential equations, PAE-Grelhas, Curved elements, Finite Element Method.

1 Introduction

Structural analysis is one of the key steps in the development of any construction project. According to Martha [1], it includes planning, designing, constructing, and ensuring the maintenance of structural systems intended for transportation, housing, work, and leisure. Once the application of numerical methods on computers became feasible, their use in structural analysis became common – and, in many cases, indispensable – for professionals in the field. Furthermore, Rigo [2] notes that with the rapid evolution of computers, increasingly efficient algorithms have been developed to faithfully represent the behavior of structures. In this regard, this development is directly related to the creation of software capable of optimizing the structural analysis process in construction projects.

Therefore, the goal here is centered on the calculation of internal forces, rotations, and displacements distributed along the entire length of grillages composed of straight and curved elements, using differential equations. In the field of civil construction, Martha [1] notes that grillage analysis can be used to model bridge decks or building floors. Accordingly, the results were compared with those obtained from structural analysis software, and the resulting expressions were incorporated into the free software PAE-Grelhas, which is a FEM-based program for linear analysis of grillages with both circumferential arc-shaped and straight elements. Its first version was developed by Lima and Soares [3], then received a Graphical User Interface (GUI) by Morais [4], and was later translated into Python and improved to account for a Gaussian Quadrature by Rego et al. [5].

2 Methodology

2.1 Sign convention

Initially, a literature review was carried out to gather more information on grillage elements. Particular attention was given to the sign conventions used for internal forces, which appear to vary across different authors — especially in the case of torsion. Additionally, some of them choose to omit the signs for bending in certain cases when plotting diagrams for grillage structures. For example, Sussekind [6] and Souza and Antunes [7] present cases in which bending moment signs are omitted, and their conventions for torsion also differ. When evaluating the relationship between the global and local coordinate axes, this work adopts the positive direction for torsion as proposed by the former, while assuming a bending moment convention opposite to that commonly used for 2D frames. In other words, in this convention, the bending moment is considered positive when it causes tensile stresses in the upper fibers of the structure. Figure 1 shows the positive directions adopted for the development of the differential equations.

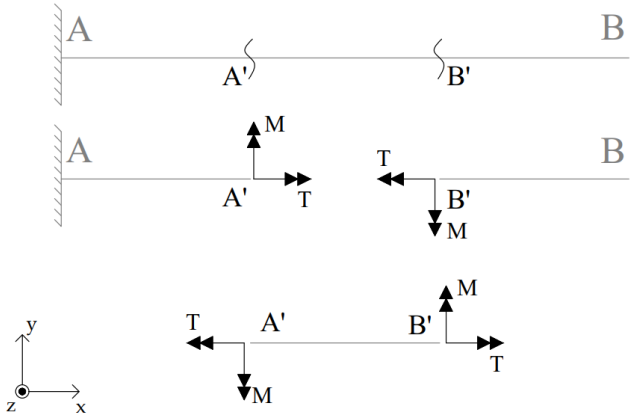


Figure 1: Positive sign convention for grillage structures in plan view.

2.2 Straight element

Figure 2 shows a generic grillage structure subjected to numerous loads in plan view, where L is the length of the element; β is the angle between the global x-axis and the local x'-axis; q is a uniformly distributed load; P is a concentrated load; and M and T are bending and torsional moment loads, respectively.

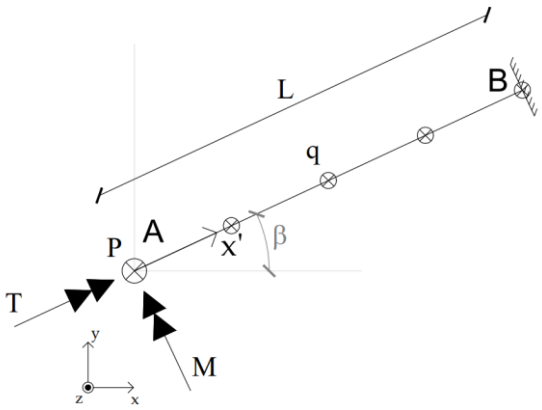


Figure 2: Straight grillage structure subjected to various loads and fixed at one end, shown in plan view.

The equations for the internal forces $M_{AB}(x')$ and $T_{AB}(x')$ are expressed in a general form, in accordance with the sign convention adopted in Fig. 1. The rotations in the global axis are calculated from the integral of these two components, as shown in eq. (1) and eq. (2).

$$\frac{d(\theta_x(x'))}{dx'} = \left\{\left[\frac{T_{AB}(x')}{G_c \cdot I_t} \cdot cos(\beta)\right] + \left[-\frac{M_{AB}(x')}{E_c \cdot I_f} \cdot sin(\beta)\right]\right\} \tag{1}$$

$$\frac{d\left(\theta_y(x')\right)}{dx'} = \left\{\left[\frac{T_{AB}(x')}{G_c \cdot I_t} \cdot sin(\beta)\right] + \left[\frac{M_{AB}(x')}{E_c \cdot I_f} \cdot cos(\beta)\right]\right\} \tag{2}$$

Both equations incorporate essential material and geometric parameters for the determinations of rotations, which are E_c, G_c, I_f and I_t (Young's modulus, shear modulus, moment of inertia, and polar moment of inertia, respectively). The integration constants can be determined using the specific boundary conditions of the analyzed structure. For the example in Fig. 2, it is assumed that $\theta_x(L) = 0$ and $\theta_y(L) = 0$ as expected for a fixed support. Therefore, $v_z(x')$ can be obtained from eq. (3), using the boundary conditions $v_z(L) = 0$ to determine its constant.

$$\frac{d(v_z(x'))}{dx'} = \{[\theta_x(x') \cdot sin(\beta)] - [\theta_y(x') \cdot cos(\beta)]\} \tag{3}$$

2.3 Curved element

As presented for the straight element, consider a generic circumferential arc-shaped grid element, shown in Fig. 3, where β is the angle between the global x-axis and the local x'-axis, which is tangent to the curve; R is the radius of the arc; α is the angular variable, measured counterclockwise up to a generic section S_1; q is a uniformly distributed load; P is a concentrated load; and M and T are bending and torsional moment loads, respectively.

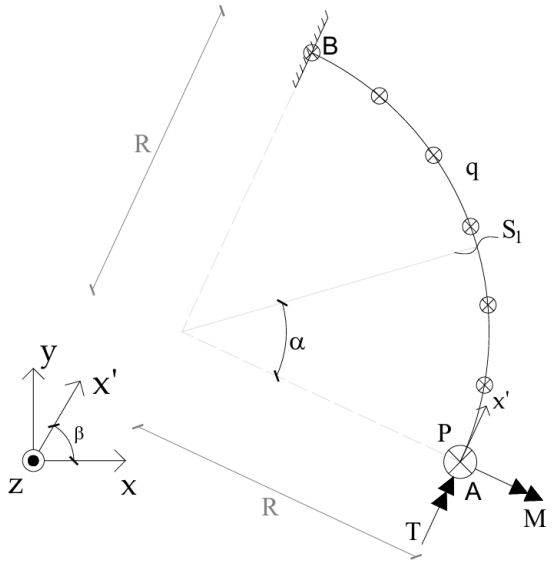


Figure 3: Curved grillage structure subjected to various loads and fixed at one end, shown in plan view.

Once again, the equations for the internal forces $M_{AB}(\alpha)$ and $T_{AB}(\alpha)$ are displayed in a general form, following the sign convention adopted in Fig. 1, which is essential for obtaining the rotations defined in eq. (4) and eq. (5).

$$\frac{d(\theta_x(\alpha))}{d\alpha} \cdot \frac{1}{R} = \left\{\left[\frac{T_{AB}(\alpha)}{G_c \cdot I_t} \cdot cos(\alpha + \beta)\right] + \left[-\frac{M_{AB}(\alpha)}{E_c \cdot I_f} \cdot sin(\alpha + \beta)\right]\right\} \tag{4}$$

$$\frac{d\left(\theta_y(\alpha)\right)}{d\alpha} \cdot \frac{1}{R} = \left\{\left[\frac{T_{AB}(\alpha)}{G_c \cdot I_t} \cdot sin(\alpha + \beta)\right] + \left[\frac{M_{AB}(\alpha)}{E_c \cdot I_f} \cdot cos(\alpha + \beta)\right]\right\} \tag{5}$$

The constants are calculated from boundary conditions, which vary according to the structures' support. The deflected shape is given by eq. (6).

$$\frac{d(v_z(\alpha))}{d\alpha} \cdot \frac{1}{R} = \{[\theta_x(\alpha) \cdot sin(\alpha + \beta)] - [\theta_y(\alpha) \cdot cos(\alpha + \beta)]\} \tag{6}$$

2.4 General equation

Based on the previously presented differential relations, a general equation was formulated for implementation in the code, aiming to calculate both rotations and the deflected shape. The required parameters are extracted from the input and output data of PAE-Grelhas 2.0 [5].

For the straight element, it includes $\theta_x(0) = \theta_{x_i}$, where θ_{x_i} represents the first node rotation about the global x-axis; $\theta_y(0) = \theta_{y_i}$, that corresponds to the first node rotation about the global y-axis; and $v_z(0) = v_{z_i}$, with v_{z_i} being the vertical displacement at the first node. In addition, parameters such as q and L, E_c, G_c, I_f and I_t, as well as M_i, T_i and C_i, which are first node bending, torsion, and shear respectively, are also employed. Due to its length, the resulting expression could not be included in this paper. For the curved element, similar parameters were used, with the addition of the radius R and the angular variable α. Once again, the resulting expression is too lengthy to be included in this paper.

3 Results

Figure 4 presents an example of a grillage structure to be calculated using LESM 3.0 [8], PAE-Grelhas 2.0 [5], which only include nodal solution, and its improved version with the deflected shape obtained from the generic equation.

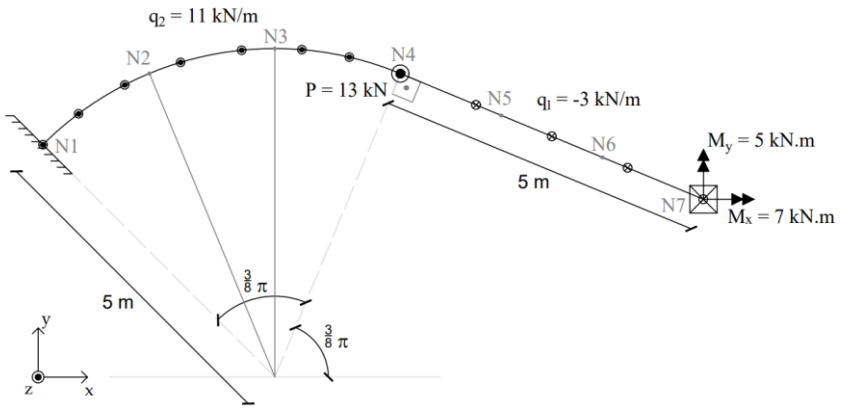


Figure 4: Grillage structure used as an example.

The structure in Fig. 4 was modeled using LESM 3.0 [8], where the straight-axis element is defined by nodes N4, N5, N6, and N7. It's worth noting that the current version of the software does not support curved elements, making it necessary to discretize the arc into a number of straight segments (it was adopted 27 for this paper), including nodes N1, N2, N3, and N4. For PAE-Grelhas 2.0 [5], only the nodes highlighted in Fig. 4 were necessary. Finally, the improved version containing the general equation required only nodes N1, N4, and N7, since any node belonging to the structure can be determined through the proper use of the variables x' and α. Table 1 shows the material and geometric properties of the cross-section.

Table 1. Cross-section material and geometric properties

Material properties		Geometric properties	
E_c	G_c	I_f	I_t
$2.0 \cdot 10^7$ kN/m²	$1.0 \cdot 10^7$ kN/m²	$7.2 \cdot 10^{-3}$ m⁴	$7.526 \cdot 10^{-3}$ m⁴

Table 2 presents the results obtained for the global vertical displacement of the structure's nodes, as computed by each software.

Table 2. Comparison of results

Nodes	Coordinates (m)		Displacements (mm)			Rotation-x (rad)			Rotation-y (rad)		
	x	y	PAE-Grelhas 3.0	LESM 3.0	PAE-Grelhas 2.0	PAE-Grelhas 3.0	LESM 3.0	PAE-Grelhas 2.0	PAE-Grelhas 3.0	LESM 3.0	PAE-Grelhas 2.0
1	-5.44895	-1.08386	0.00	0.00 **(0.00%)**	0.00 **(0.00%)**	0.00	0.00 **(0.00%)**	0.00 **(0.00%)**	0.00	0.00 **(0.00%)**	0.00 **(0.00%)**
2	-3.82683	0.0	1.4289	1.4285 **(0.03%)**	1.4289 **(0.00%)**	0.000837	0.000837 **(0.00%)**	0.000837 **(0.00%)**	-0.00094	-0.00094 **(0.05%)**	-0.00094 **(0.01%)**
3	-1.913417	0.38061	3.6797	3.6792 **(0.02%)**	3.6799 **(0.01%)**	0.00144	0.00144 **(0.00%)**	0.00144 **(0.00%)**	-0.00088	-0.00088 **(0.04%)**	-0.00088 **(0.01%)**
4	0.00	0.00	4.2604	4.2596 **(0.02%)**	4.2604 **(0.00%)**	0.00189	0.00189 **(0.00%)**	0.00189 **(0.00%)**	-0.00039	-0.00039 **(0.01%)**	-0.00039 **(0.04%)**
5	1.539799	-0.63781	3.2664	3.2654 **(0.03%)**	3.2661 **(0.01%)**	0.00215	0.00215 **(0.00%)**	0.00215 **(0.01%)**	-0.00004	-0.00004 **(0.35%)**	-0.00004 **(0.24%)**
6	3.079598	-1.275611	1.7596	1.7591 **(0.03%)**	1.7595 **(0.01%)**	0.002325	0.002325 **(0.01%)**	0.002325 **(0.00%)**	0.00012	0.00012 **(0.24%)**	0.00012 **(0.13%)**
7	4.619398	-1.913417	0.00	0.00 **(0.00%)**	0.00 **(0.00%)**	0.002459	0.002459 **(0.01%)**	0.002459 **(0.00%)**	0.00017	0.00017 **(0.18%)**	0.00017 **(0.01%)**

Note: Error values are given in parentheses.

4 Conclusions

PAE-Grelhas 3.0, presented in this study, provides a reliable representation of the elastic behavior of grillage structures while facilitating the data input process in comparison to the previous version. This enhancement stems from the fact that the program requires only the initial and final nodes of each element to compute the structure's deflected shape. Future developments will incorporate graphical functionalities, including the representation of the structure in its original and deformed states.

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