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Mathematical modeling of a space tether considering the transition from the slack cable configuration to the stretched cable configuration

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Abstract. This paper proposes a mathematical model for the dynamics of a space tether in the unstretched configuration (cable deformed but not stretched; also named slack cable configuration), in the stretched configuration (cable deformed and stretched) and considers the transition from one configuration to the other. Hamilton's principle is applied to obtain the governing equations of motion and boundary conditions for both configurations. Constraint equations are presented that determines the transition from the slack configuration to the stretched configuration and vice-versa. External forces applied to the masses (satellites) at the ends of the space tether are responsible for the stretching of the cable, which is the desired configuration. The ultimate goal of this research is that in the stretched configuration, a desired value for the tension developed in the cable must be achieved and maintained so that the space-tether operation is satisfactory.

Keywords: stretched cable, slack cable, constraint equation, space tether, mathematical modeling.

1 Introduction

This work presents the analytical development of the equations of motion of a tethered satellite system (TSS). It is composed of two spacecrafts (satellites) interconnected by a long cable or tether. This tether, often made of ultra-high-strength fiber, can range from a few meters to dozens of kilometers long. Applications of this kind of systems may be connected to electrodynamic propulsion, momentum exchange, artificial gravity, and deployment of sensors or antennas - to name just a few [1, 2, 3].
Mathematical modelling of cables and space tethers can be found in literature and it is very hard to find two authors that uses the same mathematical model [4, 5, 6, 7], regardless of the intended applications. There are excellent reviews on mathematical modelling of space tethers [8, 9]. There are also excellent books on mathematical modelling, control and dynamics of space tethers [10] (for example, for mathematical modelling and dynamics).
The mathematical model developed here considers the TSS in both the unstretched (or slack) configuration and in the stretched configuration, as well as the transition from one configuration to the other. Studies of this type are rarely found in the literature. Hamilton's principle is applied to obtain the governing equations of motion and boundary conditions. It is postulated the existence of external forces applied to the satellites at the ends of the space tether, responsible for the stretching of the cable.
The procedures for the numerical integration of the governing equations of motion of the cable and satellites are discussed but numerical results are not presented here. These procedures involve a combination of the method of finite differences for the discretization of the cable and the numerical integrator named fourth-order Runge-Kutta for the numerical integration of the complete set of governing equations of motion.

2 Geometric model and position vectors

Consider Figure 1.

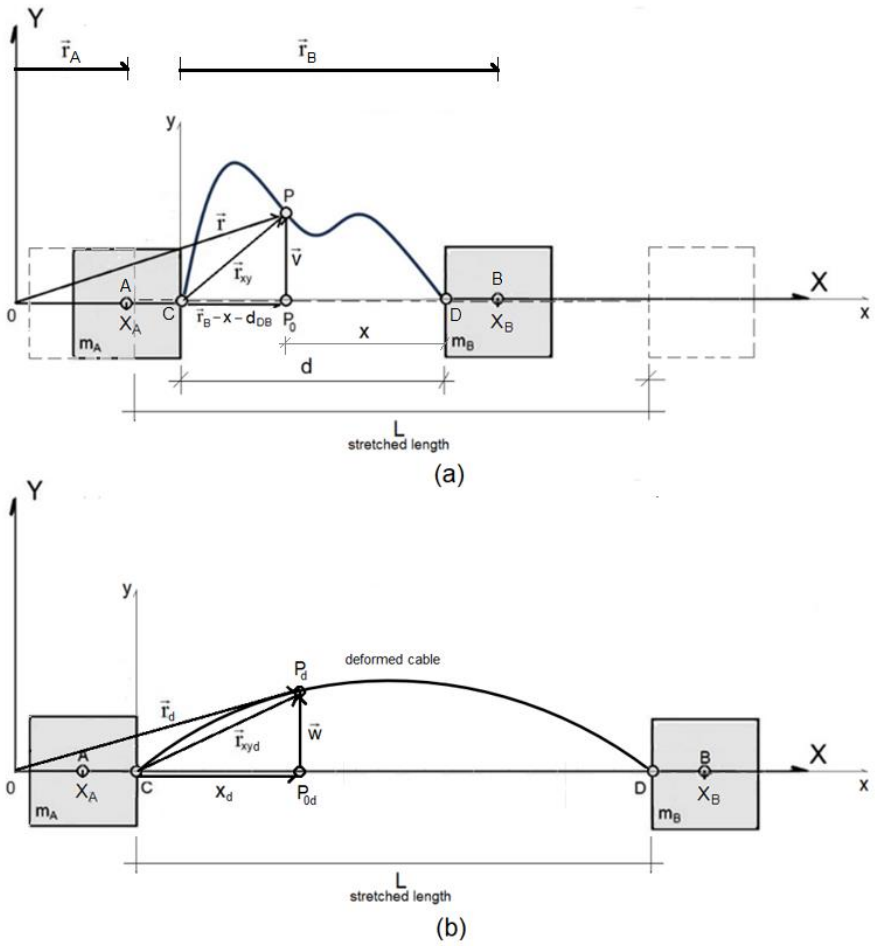


Figure 1 – The system to be investigated: (a) unstretched and (b) stretched condition (u<<<L).

Figure 1 presents the geometric models that will be used for each of the mathematical models developed in this article. Each model will be obtained separately.

In Figure 1(a) and 1(b), XY is the inertial reference frame (versors $\vec{i}$ and $\vec{j}$) and xy is the moving reference frame (versors $\vec{i}_0$ and $\vec{j}_0$).

2.1 Slack cable

In Figure 1(a):

$$\vec{r}_A(t) = x_A(t)\,\vec{i} \tag{1}$$

$$\vec{r}_B(t) = x_B(t)\,\vec{i}_0 = x_B(t)\,\vec{i} \tag{2}$$

$$\vec{v}(x,t) = v(x,t)\,\vec{j}_0 = v(x,t)\,\vec{j} \tag{3}$$

$$\vec{r}(x,t) = \vec{r}_A(t) + \vec{r}_{AC} + \vec{r}_{xy}(x,t) = \vec{r}_A(t) + \vec{r}_{AC} + \vec{r}_B(t) - \vec{x} - \vec{r}_{DB} + \vec{v}(x,t)$$

$$= \left(x_A(t) + d_{AC} + x_B(t) - x - d_{DB} \right)\vec{i} + v(x,t)\vec{j} \tag{4}$$

and

$$d = x_B(t) - d_{DB} \tag{5}$$

For this system:

$$0 < d \leq L \tag{6}$$

Therefore, the constraint equation for the system in Figure 1(a) is given by:

$$0 < x_B(t) - d_{DB} \leq L \tag{7}$$

When d = L (the maximum value for d according to Equation (6)) one has v(x, t) = 0.

2.2 Stretched cable

In Figure 1(b):

$$\vec{r}_{Ad}(t) = x_{Ad}(t)\,\vec{i} \tag{8}$$

$$\vec{r}_{Bd}(t) = x_{Bd}(t)\,\vec{i} \tag{9}$$

$$\vec{w}(x, t) = w(x, t)\,\vec{j}_0 = w(x, t)\,\vec{j} \tag{10}$$

$$\vec{r}_d(x, t) = \vec{r}_{Ad}(t) + \vec{r}_{AC} + \vec{r}_{xyd}(x, t) = \vec{r}_{Ad}(t) + \vec{r}_{AC} + \vec{x}_d + \vec{w}(x, t) = \left(x_{Ad}(t) + d_{AC} + x_d \right)\vec{i} + w(x, t)\vec{j} \tag{11}$$

and

$$L = x_{Bd}(t) - x_{Ad}(t) - d_{AC} - d_{DB} \tag{12}$$

Let L_d be the deformed length of the stretched cable. For this problem one considers:

$$L \cong L_d \tag{13}$$

Therefore, the constraint equation for the system in Figure 1(b) is given by:

$$x_{Bd}(t) - x_{Ad}(t) - d_{AC} - d_{DB} \geq L \tag{14}$$

When v(x, t) = 0, the cable is stretched and one starts to compute the tension on the cable. When the tension on the cable is equal to zero, one starts again to compute v(x, t).
From now on, the parentheses indicating the dependence of the variables on x and t will be removed.

3 Kinetic energy of the system

3.1 Slack cable

The kinetic energy of the system in Figure 1(a) is given by:

$$T = \frac{1}{2} m_1 \left|\dot{\vec{r}}_A\right|^2 + \frac{1}{2} m_2 \left|\dot{\vec{r}}_B\right|^2 + \frac{1}{2} \int_0^d \rho A_0 \left|\dot{\vec{r}}\right|^2 dx \tag{15}$$

where d is presented in Figure 1(a) and in Equation (5), ρ is the density of the cable material, m_1 is the first (A) satellite mass, m_2 is the second (B) satellite mass and A0 is the cable cross-section area. In Equation (15), the dot over the variables denotes differentiation with respect to the time t.
Using Equations (1) to (4), Equation (15) can be rewritten as:

$$T = \frac{1}{2} m_1 \dot{x}_A^2 + \frac{1}{2} m_2 \dot{x}_B^2 + \frac{1}{2} \int_0^d \rho A_0 \left(\dot{x}_A^2 + \dot{x}_B^2 + 2\dot{x}_A \dot{x}_B - 2\dot{x}_A \dot{x} - 2\dot{x}_B \dot{x} + \dot{x}^2 + \dot{v}^2 \right) dx \quad (16)$$

3.2 Stretched cable

The kinetic energy of the system in Figure 1(b) is given by:

$$T = \frac{1}{2} m_1 \left| \dot{\vec{r}}_{Ad} \right|^2 + \frac{1}{2} m_2 \left| \dot{\vec{r}}_{Bd} \right|^2 + \frac{1}{2} \int_0^L \rho A_0 \left| \dot{\vec{r}}_d \right|^2 dx \quad (17)$$

Using Equations (8) to (11), Equation (17) can be rewritten as:

$$T = \frac{1}{2} m_1 \dot{x}_{Ad}^2 + \frac{1}{2} m_2 \dot{x}_{Bd}^2 + \frac{1}{2} \int_0^L \rho A_0 \left(\dot{x}_{Ad}^2 + 2\dot{x}_{Ad} \dot{x}_d + \dot{x}_d^2 + \dot{w}^2 \right) dx \quad (18)$$

Area A0 relates to the deformed (stretched) cable cross-section area, A, according to:

$$A = \frac{A_0}{1 + \varepsilon} \quad (19)$$

where ε is the total axial strain of the cable.

4 The total axial strain of the cable

4.1 Slack cable

In the case presented in Figure 1(a), the cable is not under internal tension and therefore does not present axial deformation.

4.2 Stretched cable

Consider Figure 2 related to Figure 1(b).

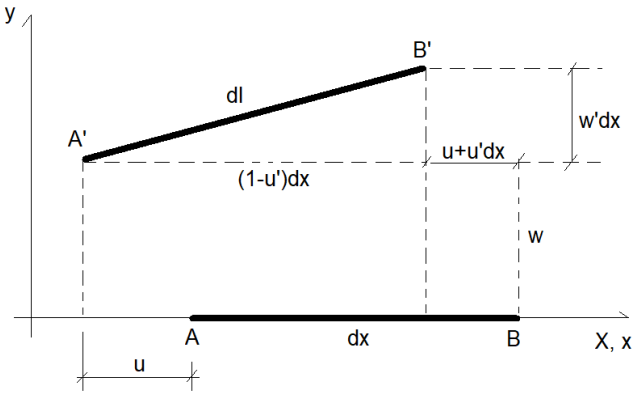


Figure 2. An infinitesimal segment of the cable presenting the undeflected (A, B) and deflected (A', B') configurations

In Figure 2, $u' = \dfrac{\partial u}{\partial x}$ and $w' = \dfrac{\partial w}{\partial x}$. According to this figure:

$$dl = \sqrt{\left(1 - \frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial w}{\partial x}\right)^2 dx^2} = \sqrt{\left(1 - \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\, dx$$

The axial strain of the cable, ε, is therefore given by:

$$\varepsilon = \frac{dl - dx}{dx} = \frac{\sqrt{\left(1 - \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2}\, dx - dx}{dx} = \sqrt{\left(1 - \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2} - 1$$

$$\varepsilon = \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \tag{20}$$

5 Potential energy of the system: strain energy of the cable

5.1 Slack cable

Considering that the cable is not stretched (subscript ns), one considers here that the cable is not under internal tension. Therefore, it does not store strain energy. Then:

$$V_{ns} = 0 \tag{21}$$

5.2 Stretched cable

Considering that the cable is stretched (subscript s), the strain energy of the cable is given by:

$$V_s = \frac{1}{2} EA \int_0^L \varepsilon^2 dx \tag{22}$$

where E is the Young modulus of the cable material. Substituting Equation (20) in Equation (22) and considering E and A as constant quantities results:

$$V = \frac{1}{2} EA \int_0^L \left\{ \sqrt{1 + \left(\frac{\partial w}{\partial x}\right)^2} - 1 \right\}^2 dx \tag{23}$$

6 Work of nonconservative forces

The non-conservative forces acting on this system are applied by the satellites and are shown in Figure 3.

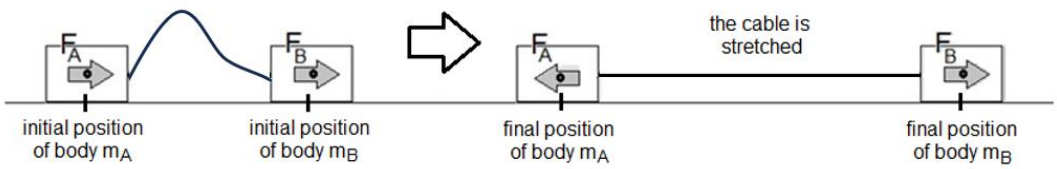


Figure 3. The control forces acting on the system: initial configuration and desired final configuration

The work of the nonconservative forces for the slack cable and for the stretched cable is given respectively by:

$$\begin{aligned} W_{nc} &= \vec{F}_A \cdot \vec{x}_A + \vec{F}_B \cdot \vec{x}_B \\ W_{nc} &= \vec{F}_A \cdot \vec{x}_{Ad} + \vec{F}_B \cdot \vec{x}_{Bd} \end{aligned} \tag{24}$$

7 The Extended Hamilton's principle

The Extended Hamilton's principle states that (no external forces considered up to here):

$$\delta \int_{t_1}^{t_2} (T - V + W_{nc})\, dt = 0 \tag{25}$$

In Eq. (25), $\delta(t_1) = \delta(t_2) = 0$.

7.1 Slack cable

Substituting Equations (16), (21) and (24) into Equation (25) and considering ρ and A, the resulting governing equations of motion are given by:

$$\left(m_1 - \rho A d_{DB} + \rho A x_B\right)\ddot{x}_A + \rho A\left(x_B - d_{DB}\right)\ddot{x}_B = F_A \tag{26}$$

$$\rho A\left(x_B - d_{DB}\right)\ddot{x}_A + \left(m_2 - \rho A d_{DB} + \rho A x_B\right)\ddot{x}_B = F_B \tag{27}$$

$$\rho A \ddot{v} = 0 \tag{28}$$

and the boundary conditions

$$v(0, t) = v(L, t) = 0 \tag{29}$$

Equation (26) is the governing equation for the variable x_A, Equation (27) is the governing equation for the variable x_B and Equation (28) is the governing equation for the variable v. The forces F_A and F_B are the forces whose purpose is to keep the cable stretched. These forces will be replaced in future developments by a control law with the aim of keeping the cable stretched and under a desired tension. In Equations (26) and (27), d is expanded using Equation (5).

7.2 Stretched cable

Substituting Equations (18), (23) and (24) into Equation (25) and considering ρ and A, the resulting governing equations of motion are given by:

$$\left(m_1 + \rho A_0 L\right)\ddot{x}_{Ad} = F_A \tag{30}$$

$$m_2 \ddot{x}_{Bd} = F_B \tag{31}$$

$$\rho A_0 \ddot{w} - E A\left[\frac{\left(1 + w'^2\right)\sqrt{1 + w'^2} - 1}{\left(1 + w'^2\right)\sqrt{1 + w'^2}}\right] w'' = 0 \tag{32}$$

and the boundary conditions

$$w(0, t) = w(L, t) = 0 \tag{33}$$

Equation (30) is the governing equation for the variable x_{Ad}, Equation (31) is the governing equation for the variable x_{Bd} and Equation (32) is the governing equation for the variable w. The forces F_A and F_B are the forces whose purpose is to keep the cable stretched. These forces will be replaced in future developments by a control law with the aim of keeping the cable stretched and under a desired tension.

8 Tension force in the cable

8.1 Slack cable

In the analysis here there is no tension in the cable in the unstretched condition.

8.2 Stretched cable

The internal tension force in the cable, τ, is given by:

$$\tau = \mathrm{E}\,\mathrm{A}\varepsilon \tag{34}$$

or, using Equation (20):

$$\tau = \mathrm{E}\,\mathrm{A}\left(\sqrt{1+\mathrm{w}'^2}-1\right) \tag{35}$$

9 Scheme for the numerical integration

Figure 4 presents a scheme regarding the steps to be followed for the interaction between the two sets of governing equations of motion for the space tether in the numerical integration.

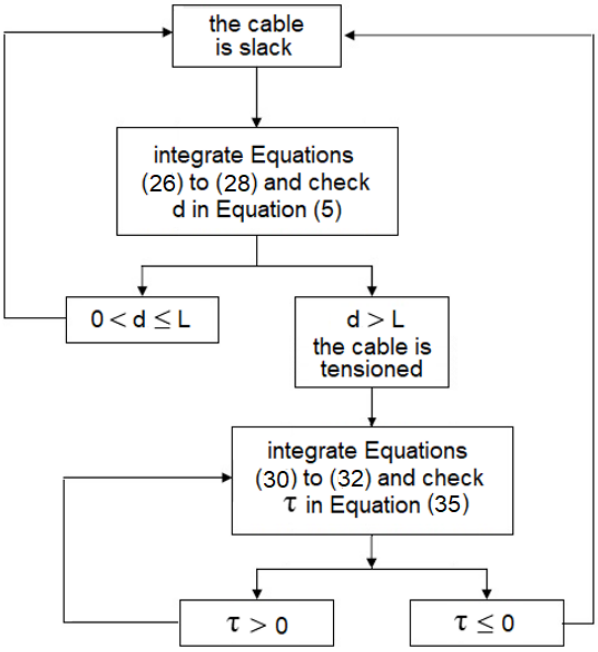


Figure 4. Scheme for the numerical integration of the governing equation of motion for the slack-stretched tether

10 Numerical integration of the governing equations of motion

10.1 Slack cable

In order to perform the numerical integration of the governing equations of motion, the cable as seen in Figure 1(a) is divided into elements according to Figure 5. The distance d(t) is divided in 8 elements (or 9 nodes). The element size changes with time and is given by d(t)/8.

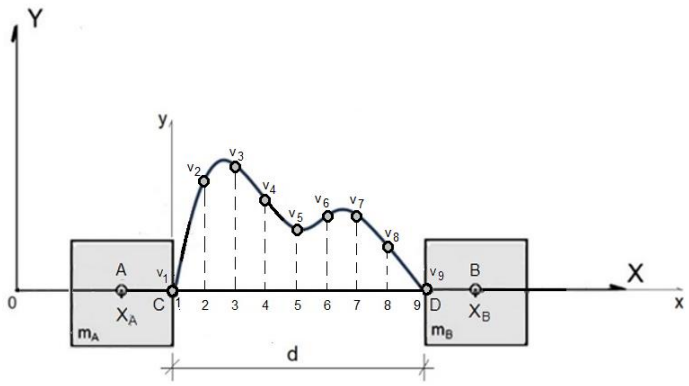


Figure 5. Division of the distance d in elements for the discretization of the cable

Decoupling Equations (26) and (27), the system of equations (26) to (28) is written as:

$$\ddot{x}_A = \left(\frac{c}{ac-b^2}\right)F_A + \left(\frac{-b}{ac-b^2}\right)F_B \tag{36}$$

$$\ddot{x}_B = \left(\frac{-b}{ac-b^2}\right)F_A + \left(\frac{a}{ac-b^2}\right)F_B \tag{37}$$

$$\ddot{v} = 0 \tag{38}$$

where

$$a = \left(m_1 - \rho A d_{DB} + \rho A x_B\right)$$

$$b = \rho A\left(x_B - d_{DB}\right)$$

$$c = \left(m_2 - \rho A d_{DB} + \rho A x_B\right)$$

$$d = \rho A$$

According to the discretization depicted in Figure 5, the final set of governing equations of motion for the slack cable to be numerically integrated is finally given by:

$$\ddot{x}_A = \left(\frac{c}{ac-b^2}\right)F_A + \left(\frac{-b}{ac-b^2}\right)F_B \tag{39}$$

$$\ddot{x}_B = \left(\frac{-b}{ac-b^2}\right)F_A + \left(\frac{a}{ac-b^2}\right)F_B \tag{40}$$

$$\ddot{v}_k = 0, \quad k = 2, 3, \ldots, 8 \tag{41}$$

The position of each node in the x direction is given by Equations (39) and (40).

10.2 Stretched cable

In order to apply the finite difference method, the cable as seen in Figure 1(b) must first be divided into elements according to Figure 6.

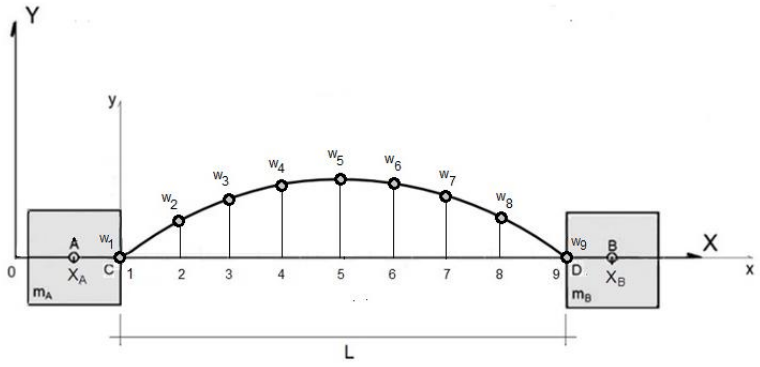


Figure 6. Division of the length L in elements for the application of the finite difference method

The length L is divided in 8 elements (or 9 nodes). The element size is h = L/8.
In Equation (32), two derivatives are discretized using finite difference method: w' and w''.
Considering nodes 2 to 8, the central difference formula using a five-point stencil is considered. The formulas for the first and second derivatives, respectively, are given by:

$$\frac{\partial w(x_i,t)}{\partial x} = \frac{-w(x_i+2h,t)+8w(x_i+h,t)-8w(x_i-h,t)+w(x_i-2h,t)}{12h} \tag{42}$$

$$\frac{\partial^2 w(x_i,t)}{\partial x^2} = \frac{-w(x_i+2h,t)+16w(x_i+h,t)-30w(x_i,t)+16w(x_i-h,t)-w(x_i-2h,t)}{12h} \tag{43}$$

For nodes 2 and 8, the boundary conditions are used in order to eliminate the points located outside the beam (related to w_0 and w_{10}). Using Equations (42) and (43), one writes, for $k = 2,3,\ldots,8$:
Node *k*:

$$\frac{\partial w_k}{\partial x} = \frac{-w_{k+2}+8w_{k+1}-8w_k+w_{k-2}}{12h} \tag{44}$$

$$\frac{\partial^2 w_k}{\partial x^2} = \frac{-w_{k+2}+16w_{k+1}-30w_k+16w_{k-1}-w_{k-2}}{12h} \tag{45}$$

According to the discretization depicted in Figure 6, the final set of governing equations of motion for the stretched cable to be numerically integrated is finally given by:

$$(m_1+\rho A_0 L)\ddot{x}_{Ad} = F_A \tag{46}$$

$$m_2\ddot{x}_{Bd} = F_B \tag{47}$$

$$\rho A_0\ddot{w}_k - EA\left[\frac{\left(1+w_k'^{\,2}\right)\sqrt{1+w_k'^{\,2}}-1}{\left(1+w_k'^{\,2}\right)\sqrt{1+w_k'^{\,2}}}\right]w_2'' = 0, \qquad k = 2,3,\ldots,8 \tag{48}$$

For i=2 to 8, the discretized variables w_i' and w_i'' are given by Equations (44) and (45).

11 Conclusions

This article presents the mathematical modeling of a dynamic system to be used in the study of a space tether. This model is developed using the extended Hamilton's principle based on the system's kinetic and potential energies and the work developed by external forces. The innovation in this study is that the modeling involves the slack

phase of the cable and its transition to the stretched phase. In this sense, a different model is developed for each of these phases as well as the transition rules from one phase to the other. The cable is appropriately discretized and the resulting equations are prepared for numerical integration. The finite difference method and the fourth-order Runge-Kutta method will be used in the future, respectively, for these purposes. This is a first model to be revisited and improved in the future.

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