XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGENEERING

Stiffness parameter identification of a planar structural element based on the Modified Constitutive Relation Error

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Abstract. Visual inspection techniques for structural damage detection are inherently limited: damage can be concealed, too minor for visual detection, or situated in inaccessible areas such as the interior of a nuclear reactor. To overcome these challenges, model-based approaches grounded in the solution of inverse problems have gained prominence. In particular, finite element model updating using static or dynamic response data has emerged as a powerful strategy to identify and quantify damage. By calibrating numerical models to reflect observed structural behavior more accurately, these methods support more informed maintenance planning and targeted intervention. This study presents a numerical strategy for identifying the structural parameters of a steel plate by minimizing a modified Constitutive Relation Error (MCRE) functional. Starting from a reference measure, the method seeks admissible mechanical fields and localizes regions with high errors. Once this iterative step converges, parameter updating is initiated to estimate the material properties accurately. The MCRE framework is based on the partitioning of the governing equations into a reliable set—where all equations are strictly satisfied—and an unreliable set, which encompasses potential deviations in the constitutive law. In the first phase, the admissible fields are obtained solving the global balance equations using a quadratic energy function that respects both Neumann's and Dirichlet's boundary conditions. In the second phase, the material parameters are identified through the minimization of a cost function that incorporates the previously obtained admissible fields and the reference measures. The Cast3M open-source toolbox for finite element computation was chosen to implement MCRE algorithm. The results obtained in this study demonstrate that the proposed numerical strategy was effective in accurately identifying both the stiffness parameters and the location of the defect in the plate.

Keywords: Modified Constitutive Relation Error, Optimization, Planar Structures, Finite Element Model Updating, Cast3M

1 Introduction

Modern infrastructures such as bridges, buildings, dams, nuclear power plants and oil platform operates under various complex loads, environmental conditions and operational demands. The performance and integrity of these structures over time is critical to ensure safety and reliability. Strategies to minimize failure, reduce downtime and ensure functionality have emerged as a priority in engineering. Among them, structural health monitoring (SHM) systems have allowed to observe, acquire periodically spaced measurements and extract appropriate damage-sensitive features and analyze them to diagnose the ongoing state of system health continuously over time [1]. One of the key challenges in SHM lies on the accurate interpretation of the measurements to identify and quantify damage. Models that precisely capture real structural behavior enable more effective interventions and maintenance. Needless to say that only raw sensor data is not sufficient to provide a complete understanding of the structural behavior. In this context, high fidelity numerical models, such as finite element models (FEM) are required.

FEM plays an important role in structural engineering, mainly in the solution of forward problem, where given a set of material properties, known geometry, boundary conditions and loads, it is predicted the structural response, whether it is static or dynamic. On the other hand, SHM lies on inverse problem, where the goal is to

improve a mathematical model of an existing structure using measurements performed on this structure [2]. This process allows to infer changes in stiffness, mass or identify damage in the structure.

In damage identification, visual inspection often faces inherent limitations: (i) damage may be hidden, (ii) it may be too small to detect visually, or (iii) it may be located in inaccessible regions (e.g., the interior of a nuclear reactor). To address these challenges, updating finite element models [3] using static or dynamic reference data has become a powerful tool for damage detection. In model updating, various classes of methods have been proposed as the direct methods, sensitivity methods or even error localization. Since inverse problems are often ill-posed, attention must be given to the conditioning of the problem. The modified constitutive relation error (MCRE) emerged in the context of the Constitutive Relation Error (CRE), an error estimator to verify the quality of finite element solutions [4].

The proposed formulation of MCRE is framed as a constrained optimization problem, in which a quadratic energy-based cost function is minimized to quantify the discrepancy between predicted and measured system responses [5]. This technique allows to choose physical quantities as input parameters such as stiffness, mass in opposition to direct methods that often rely on linear algebra principles.

Recent developments in MCRE-based methods have broadened their applicability to various engineering fields such aerospace industry [5], civil engineering [6], [7], [8], mechanical engineering [9], in transient problem [10], not limited to FEM but also to other numerical methods such as boundary element method (BEM) [11]. The modern experiment techniques, such as digital image correlation (DIC), have also been integrated into MCRE-based methods [12],[13] or using optical fiber strain measurement device to detect damage in concrete beam [14]. In the framework of MCRE-based methods, applications have been reported even in the investigation of fiber orientation in composite materials [15].

In the present study, the augment version of Constitutive Relation Error (MCRE) is used as an updating model technique. It is intended to demonstrate the fundamental idea of the technique, introduce the formulations and illustrate its efficiency to identify the stiffness parameter of a planar structural element. In this example, it is used static synthetic data from a defect region of the structure as reference measures. This application is limited to a simple case in a deterministic domain, but the key aspects presented here can be extended to more complex cases. The MCRE method is implemented using the open-source finite element software [16], which provides a flexible and efficient platform for numerical simulations, developed by the French Atomic Energy Commission (CEA).

2 CRE reference problem

Let it consider the general case of a structure within the domain $\Omega \subset \mathbb{R}^d$ with boundary $\partial\Omega = \partial_1\Omega \cup \partial_2\Omega$, where $\partial_1\Omega$ is the boundary where the displacement is prescribed and $\partial_2\Omega$ is the boundary where the surface loads F_{d1} are imposed. Herein the absence of body force in the reference problem will be considered. Let (σ, u) be a prospective solution. It's said admissible fields when the following properties are satisfied:

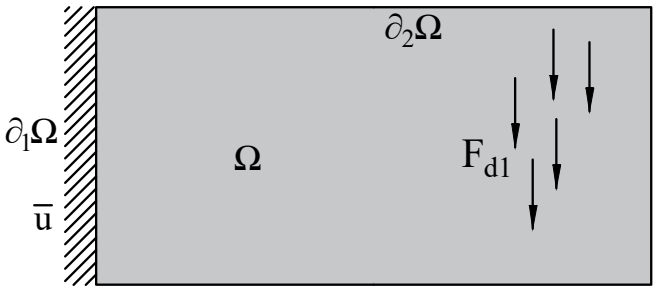


Figure 1. Reference problem

- Kinematic boundary condition:

$$u|_{\partial_1\Omega} = \overline{u} \tag{1}$$

- Equilibrium equations:

$$\sigma \cdot n|_{\partial_2\Omega} = F_{d1} \tag{2}$$

$$\int_\Omega \sigma : \varepsilon^* \mathbf{d\Omega} = \int_{\mathbf{\Omega}} \mathbf{b} \cdot \mathbf{u}^* \mathbf{d\Omega} + \int_{\mathbf{\Omega}} (\sigma \cdot \mathbf{n}) \cdot \mathbf{u}^* \mathbf{d\Omega}, \quad \forall \mathbf{u}^*|_{\partial_1 \mathbf{\Omega}} = \mathbf{0} \tag{3}$$

Although admissibility is essential, it does not imply uniqueness. It is necessary to specify the constitutive relation such that: f(σ, ε) = $\sigma - \mathbb{C}\varepsilon = 0$.

The first key principle of the CRE is splitting the mechanical equations into two groups: a reliable set where all equations must be satisfied since the information about this group is known and reliable and a less reliable set, here the constitutive relation, for the sake of do not represent the actual material behavior with unknown parameters $\underline{\theta}$. This separation is not closed but can vary depending on the problem and the analyst should it make appropriate to the case study. The reliable equations allow to define the kinematically admissible spaces $\mathcal{U}_{Ad}$ and statically admissible spaces $\mathcal{S}_{Ad}$.

As a second principle, just as important as the former, the less reliable set of equation is verified as closely as possible through the minimization of a functional. A solution is said admissible when it belongs to both spaces, i.e., (σ, $\mathbf{u}$) $\in \mathcal{S}_{Ad} \cap \mathcal{U}_{Ad}$. The core of the method lies on the residual or the discrepancy between the stress field statically admissible and the displacement field kinematically admissible. The residual is defined as:

$$E^2_{MCRE} = \underbrace{\frac{1}{2}\int_{\Omega}(\sigma - \mathbb{C}\varepsilon) : \mathbb{C}^{-\mathbf{1}}(\sigma - \mathbb{C}\varepsilon)\mathbf{d\Omega}}_{J_1} + \underbrace{\frac{\alpha}{2(1-\alpha)}\|\mathbf{u} - \tilde{\mathbf{u}}\|}_{J_2} \tag{4}$$

As it can be seen, the functional expressed by eq. (4) is made of two terms: the CRE J_1 and J_2 is the distance or gap between the measured and the calculated displacement field. E^2_{MCRE} is said to be the modified constitutive relation error functional and works also as a performance measure. This measure is quadratic thus possesses solely one global minimum and is equal to zero when the exact solution is found. The coefficient $\alpha/(1-\alpha)$ is a weighting factor [0,1] that balances the degree of confidence to the measurements. Greater values of α indicate a higher confidence in the measurements, while lower values (close to 0) suggest the opposite case. The solution of the mechanical problem, continuous form, will be the minimization as follows:

Problem 1

$$\min_{\sigma,\mathbf{u}} E^2_{MCRE} \tag{5}$$

such that $\sigma \in \mathcal{S}_{Ad}$, $\mathbf{u} \in \mathcal{U}_{Ad}$.

The problem above is for a fixed mechanical parameter θ. The next step is to find the best mechanical parameter θ by minimizing the cost function obtained from the solution fields of the problem 1. The solution found must respect both the admissibility conditions and the constitutive relation and should respect the observed measures. This second problem clearly lies on a double constrained optimization problem It can be stated as follows:

Problem 2

$$\min_{\mathbb{C}} \min_{\sigma,\mathbf{u}} E^2_{MCRE} \tag{6}$$

such that $\sigma \in \mathcal{S}_{Ad}$, $\mathbf{u} \in \mathcal{U}_{Ad}$ and $\mathbb{C}$ is admissible.

3 Discrete formulation of the problem

The reference problem as well as the CRE framework until this point was introduced in a continuous form. It is clear that to implement the method in a numerical code, it is necessary to discretize the problem. One may solve the problem using the finite element method as the case of the present study. In summary, the discretization of the problem yields to the stiffness matrix, nodal vectors such that U and V are the unknowns nodal displacements vector both related to the kinematically admissible and statically admissible fields, respectively. The discrete formulation of the MCRE to be minimized is written as:

$$\mathcal{L}(U,V,\lambda) = \frac{1}{2}(V-U)^T[K](V-U) - \lambda^T(F-[K]V) \quad + \frac{\alpha}{2\cdot(1-\alpha)}(\Pi_1 U - \bar{U})^T[G](\Pi_1 U - \bar{U}) \tag{7}$$

where $[K]$ is the global stiffness matrix, λ is the Lagrange multiplier vector that relaxes the constrains of the optimization problem, Π_1 is the projection operator (matrix) which when applied to the nodal vector U yields value

of the vector at sensor positions assuring the correspondence between the measured and calculated displacements, $\bar{U}$ is the vector of reference measures. The matrix $[G]$ is a stiffness matrix at the points where the displacement is measured and weights the contribution of each node in the cost function. The stationarity of this equation leads to the following linear system of equations:

$$[A]X = B \tag{8}$$

$$A = \begin{bmatrix} K + \dfrac{\alpha}{1-\alpha} \cdot \Pi^T G \cdot \Pi & -K \\ 0 & K \end{bmatrix} \tag{9}$$

$$X = \begin{Bmatrix} U \\ V \end{Bmatrix} \tag{10}$$

$$B = \begin{Bmatrix} \dfrac{\alpha}{1-\alpha} \cdot \Pi^T G \tilde{U} \\ F \end{Bmatrix} \tag{11}$$

$$\frac{\partial \mathcal{L}}{\partial U} = K(V-U) + \frac{\alpha}{1-\alpha}\Pi_1^T G(\Pi_1 U - \bar{U}) = 0 \tag{12}$$

$$\frac{\partial \mathcal{L}}{\partial V} = K(V-U) - K\lambda = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = F - KV = 0 \tag{14}$$

Assuming that the constitutive relation depends on θ, i.e, $\mathbb{C} = \mathbb{C}(\theta)$, the equation above remains valid for a fixed θ. In other terms, the stationary condition is function of the parameter θ.

$$F(\theta) = \mathcal{L}(U(\theta), V(\theta), \lambda(\theta), \theta) \tag{15}$$

The derivation of eq. (15) yields the gradient of the cost function with respect to the parameter θ:

$$\nabla F = \frac{\partial \mathcal{L}}{\partial U}\frac{\partial U}{\partial \theta} + \frac{\partial \mathcal{L}}{\partial V}\frac{\partial V}{\partial \theta} + \frac{\partial \mathcal{L}}{\partial \lambda}\frac{\partial \lambda}{\partial \theta} + \frac{\partial \mathcal{L}}{\partial \theta} \tag{16}$$

Since eq. (12), eq. (13), eq. (14) are equal to zero due to the stationarity of the problem, the gradient of the cost function can be simplified as follows:

$$\nabla F = \frac{1}{2}\{V-U\}^T \frac{\partial K}{\partial \theta}\{V-U\} - \lambda^T \frac{\partial K}{\partial \theta}\{V\} \tag{17}$$

The updating of the parameter θ^{i+1} is obtained from the initial guess θ_0 and by eq. (17) using the steepest gradient method, or any other optimization method, such as the conjugate gradient method, Newton's method or quasi-Newton methods. This iterative process is repeated until the convergence criterion is satisfied, which can be based whether the global MCRE is below to a certain value or even no significant change in the parameter θ is observed. From the brief explanation above, it is evident that the MCRE algorithm requires, within the framework of the finite element method, the manipulation of various matrices, whether they are stiffness matrices, mass matrices, or damping matrices. Henceforth, the toolbox [16] was chosen to implement the MCRE algorithm.

4 Numerical Example

In this section, the formulation of the MCRE is illustrated through an application. The intention is to show some features of the method as well as important ideas. Consider an ideal structure with a rectangular plate with length L = 4.0 m and height H = 3.0 m, Poisson's ratio ν = 0.3, Young's modulus E = 210 GPa, and density ρ = 7850 kg/m^3. The schematic representation of the plate is shown in Fig. 2. The plate is subjected to a vertical uniform traction of $F_1 = 2 \cdot 10^6$ N on the right side. The left side is fixed while the right, top and bottom are free to move in any direction. The objective is to identify the stiffness parameters of the plate using the MCRE method. It should be noted that the plate has a defect in the middle, which is represented by red square region in Fig. 2, characterized by a reduction of the Young's modulus, i.e., E = 10.5 GPa.

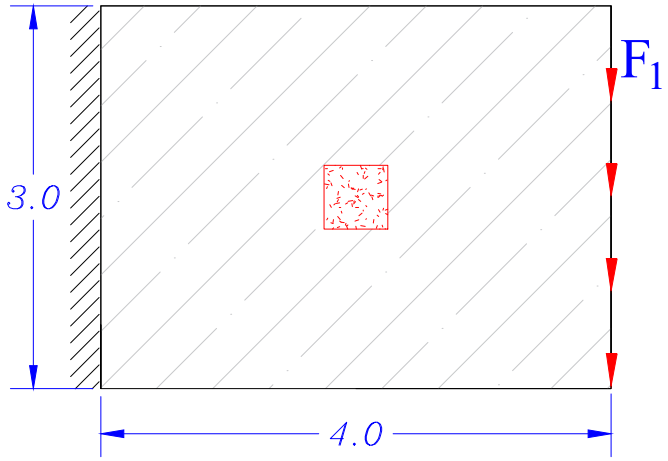


Figure 2. Plate schematic representation

The reference measures (displacements) at each node of the plate are obtained by solving the static problem using the finite element method. The first key aspect of the MCRE method to be discussed is the indication of the regions where the values of the material properties are far from the reference values and thus need to be updated. To illustrate the participation of each region in the global error, it was proposed to use a heat map, which is shown in Fig. 3. As expected the region at the middle of the plate, where the defect is located has the highest error. This is a crucial step in the MCRE method, yielded by the solution of the eq. (7) where it is obtained the admissible fields.

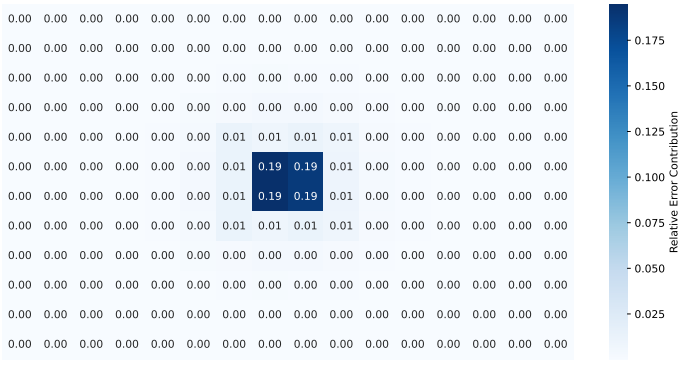


Figure 3. Contribution of each region of the structure to the global CRE

Instead of updating the stiffness parameters of the whole plate, solely the sub-zones with the highest error are updated. This brings us to the second key point of the MCRE method: the true stiffness parameters obtained by the minimization of the cost function. The evolution of the MCRE as a function of the relative stiffness parameters is shown in Fig. 4. To ensure a clear understanding of the results, and consequently of Fig. 4, the initial value of

the Young's modulus (θ_0) was set to 1.05 GPa. As the minimization progresses, the relative stiffness parameter approaches its true value, the MCRE decreases, and convergence to the reference problem becomes faster. When $\theta/\theta_0 = 1.0$, the MCRE reaches zero, indicating that the correct stiffness parameter has been identified and the solution has fully converged to the reference problem. It should be noted that the MCRE is zero only because the synthetic data were obtained numerically. When dealing with real data, the MCRE will not reach zero due to uncertainties, but it will tends to zero as demonstrated in previous works [8], [6].

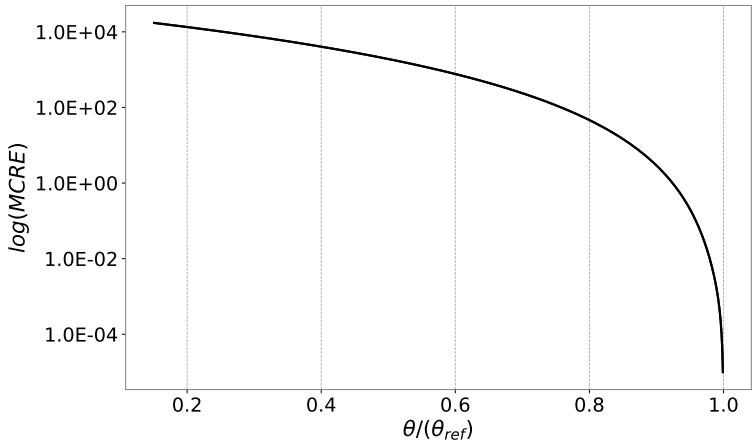


Figure 4. Modified Constitutive Relation Error (MCRE) evolution - Initial guess θ_0 = 1.05 GPa

One may observe that the MCRE also converged to zero from a different initial value of the parameter θ_0. As illustrated in Fig. 5, even if the new initial attempt (11.025 GPa) is upper to the reference value, the MCRE still converges to zero. This is a key feature of the MCRE method, which allows it to converge to the reference solution regardless of the initial guess of the parameter θ.

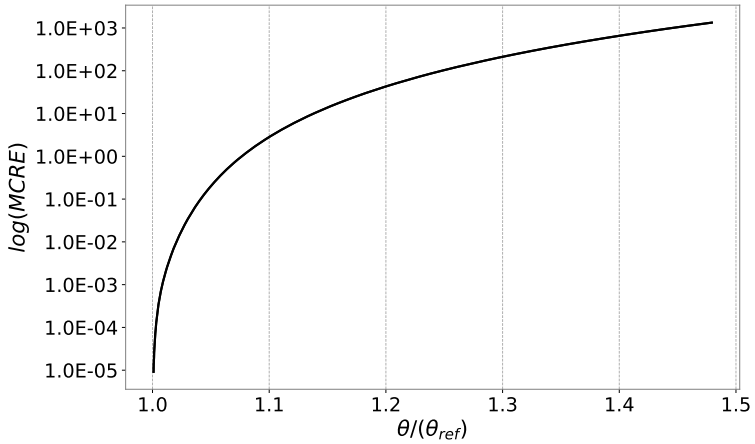


Figure 5. Modified Constitutive Relation Error (MCRE) evolution - Initial guess θ_0 = 11.025 GPa

5 Conclusions

In this paper, it was introduced the modified constitutive relation error formulation as a numerical strategy to identify the stiffness parameters of structure under static load condition. Although in this study synthetic data was used, same procedure could be applied to *in situ* measurements. The MCRE method, in summary, relies on the minimization of a energy-based cost function based on the Constitutive Relation Error (CRE) framework. In the numerical application of a planar element, it was exposed how the algorithm conducted to the localization of the defect in the plate and it provided the correct stiffness parameters of the plate in that region proving the capabilities

of this numerical identification process. Furthermore, the MCRE method may be a potential solution for structural health monitoring. Not only to detect damage but also it could be proposed an MCRE-based formulation to set up the optimal sensor placement as function of the parameters to be identified. It is far from the scope of this study to exhaust all the possibilities of the MCRE method. One of the limitations may reside on the fact that it was assumed that the constitutive relation is linear and isotropic. However, the MCRE method can be extended to nonlinear and anisotropic materials.

Acknowledgements. This study was funded by FAPESP (grant 2024/17151-6), São Paulo Research Foundation. This study was also financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil (CAPES) - Finance Code 001, Universidade Federal do Acre (UFAC) and Universidade Estadual de Campinas (UNICAMP).

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