XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGENEERING

Strain localization analysis in micromorphic media from macro and microscale perspectives

Pamela D. N. Reges[1,2], Roque L. S. Pitangueira[2], Leandro L. Silva[2]

[1]*Dept. of Civil Engineering, Federal University of Tocantins*
NS-15 Avenue, Block 109, 77001-090, Palmas/Tocantins, Brazil
pamela.daniela@uft.edu.br
[2]*Dept. of Structural Engineering, Federal University of Minas Gerais*
Antônio Carlos Avenue, 6627,31270-901, Belo Horizonte/Minas Gerais, Brazil
roque@dees.ufmg.br, leandro@dees.ufmg.br

Abstract. The study of failure conditions and failure modes at the material level within classical continua has been extensively studied in terms of a localization analysis, where discontinuous failure initiates when the second-order localization tensor or its symmetric counterpart becomes singular. Strain localization, characterized by the formation of jumps in field variables across a singularity surface, serves as a threshold for the introduction of discontinuities when damage and failure are approached from a continuous-discontinuous perspective. In these models, a continuous model simulates micro-cracking and, upon localization, strong discontinuities are introduced to simulate crack formation. Considering the multiscale nature of deformation and failure processes, specially in microstructured materials, extending continuous-discontinuous models to generalized continuum frameworks are of interest. However, the micromorphic continuum, in particular, is well know for its regularization properties due to its high level of generalization and non-local formulation. In this study, a localization indicator that overcomes the regularization effects of the micromorphic continuum is presented, enabling the identification of both the onset and direction of strong discontinuities. The micromorphic acoustic tensor is studied from macro- and microscale perspectives in an elastic degradation framework. Numerical investigations are performed using implementations developed within the open-source software INSANE.

Keywords: Micromorphic continuum; strain localization; acoustic tensor.

1 Introduction

Failure models for microstructured materials are commonly grouped into macroscopic and mesoscopic approaches [1]. Macroscopic models treat the material as homogeneous and can be subdivided into discontinuous and continuous formulations according to the representation of cracking. Discontinuous (or discrete crack) models introduce strong discontinuities in the displacement field, corresponding to sharp physical cracks [2]. Continuous (or smeared crack) models instead represent fracture as strain localization within a finite-width band, bounded by weak discontinuities or regularized to maintain smooth displacement and strain fields [3].

Hybrid continuous–discontinuous formulations combine the strengths of both, simulating micro-cracking with a continuous model and introducing discrete cracks once a critical damage threshold is reached [3–8]. Their application relies on a crack-initiation criteria, often based on the onset of strain localization or loss of ellipticity. Such conditions can be derived from acceleration wave theory, leading to localization criteria expressed through the acoustic tensor [9, 10].

In this work, the acoustic tensor is employed as a discontinuous failure criterion within the micromorphic continuum framework, following the acceleration-wave formulation of Eremeyev et al. [11]. Considering the regularization properties of the micromorphic theory, the acoustic tensor is proposed based on macro- and microscale approaches.

The paper is organized as follows: Section 2 reviews the fundamentals of micromorphic theory and damage modeling; Section 3 develops the localization analysis based on acceleration waves from macro- and microscale perspectives; Section 4 presents a finite element example under uniaxial loading to illustrate the effectiveness of the proposed approach.

2 Micromorphic elastic-degrading media

The micromorphic theory, introduced by Eringen and Şuhubi [12] and Mindlin [13], accounts for both the displacement of macroscopic material points, represented by the displacement vector u_l, and the material substructure, described by the micromotion gradient tensor $\phi_{kl} = \chi_{kl} - \delta_{kl}$, where χ_{kl} is the microdeformation tensor and δ_{kl} is the Kronecker delta. The latter considers that the microstructure can rotate, stretch, shear and, shrink. In this theory, the constitutive equations can be written

$$t_{kl} = A_{klmn}\epsilon_{mn} + E_{klmn}e_{mn}, \quad s_{kl} = E_{mnkl}\epsilon_{mn} + B_{klmn}e_{mn}, \quad m_{klm} = C_{lmknpq}\gamma_{npq} \tag{1}$$

depending on the set of linear strain measures [12, 14]

$$\epsilon_{kl} = u_{l,k} - \phi_{lk}, \qquad 2e_{kl} = \phi_{kl} + \phi_{lk}, \qquad \gamma_{klm} = \phi_{kl,m} \tag{2}$$

where t_{kl} is the stress tensor; s_{kl} is a symmetric stress tensor named micro-stress average; m_{klm} is the stress moments tensor; and A_{klmn}, B_{klmn}, C_{klmnpq}, E_{klmn} are the constitutive moduli. The construction of the constitutive equations for a linear isotropic medium within the micromorphic theory requires 18 elastic parameters, due to the 12 degrees of freedom considered (including 9 additional degrees of freedom associated with the material substructure). This challenge is addressed through a homogenization strategy, in which the macroscopic constitutive tensors are derived under the assumption of a classical continuum at the microscale. Further details on this procedure can be found in Silva et al. [15].

2.1 Scalar-isotropic damage models

Following Reges et al. [16], the micromorphic constitutive operators can be compactly written in tensorial form as

$$\Sigma_{\beta\nu} = \mathcal{E}^S_{\beta\nu\delta\psi}\Gamma_{\delta\psi}, \text{ for } \beta,\nu,\delta,\psi = 1,2,...9 \tag{3}$$

where $\Sigma_{\beta\nu}$ and $\Gamma_{\delta\psi}$ are generalized stress and strain tensors (second-order, dimension nine), and ${\mathcal{E}^S}_{\beta\nu\delta\psi}$ is the generalized secant stiffness operator assembling the four micromorphic constitutive tensors into a single fourth-order tensor. In a scalar–isotropic damage model, stiffness degradation is governed by a single damage variable $\mathscr{D}$, so that

$$\Sigma_{\beta\nu} = (1-\mathscr{D}){\mathcal{E}^0}_{\beta\nu\delta\psi}\Gamma_{\delta\psi}\,, \quad \mathcal{E}^S_{\beta\nu\delta\psi} = (1-\mathscr{D}){\mathcal{E}^0}_{\beta\nu\delta\psi} \tag{4}$$

where ${\mathcal{E}^0}_{\beta\nu\delta\psi}$ is the initial elastic operator. The generalized tangent stiffness operator is then defined

$${\mathcal{E}^t}_{\beta\nu\delta\psi} = (1-\mathscr{D}){\mathcal{E}^0}_{\beta\nu\delta\psi} - \left(\frac{\partial\mathscr{D}(\Gamma_{eq})}{\partial\Gamma_{eq}}\right){\Sigma^0}_{\beta\nu}\frac{\partial\Gamma_{eq}}{\partial\Gamma_{\delta\psi}} \tag{5}$$

where $\mathscr{D}(\Gamma_{eq})$ is a damage law that describes the evolution of the damage variable, Γ_{eq} is a generalized damage measure and ${\Sigma^0}_{\beta\nu} = {\mathcal{E}^0}_{\beta\nu\delta\psi}\Gamma_{\delta\psi}$. Extensions of the classical Simo–Ju [17], Ju [18], Marigo [19], and Mazars-Lemaitre [20] formulations to the micromorphic continuum employ the following equivalent strain definitions [16]:

$$\Gamma_{eq} = \begin{cases} \sqrt{2\psi^0} & \text{(Simo-Ju micromorphic model)} \\ \psi^0 & \text{(Ju micromorphic model)} \\ \sqrt{\frac{2\psi^0}{E}} & \text{(Marigo micromorphic model)} \\ \sqrt{\epsilon_{mn}\epsilon_{mn} + e_{mn}e_{mn}} & \text{(Mazars-Lemaitre micromorphic model)} \end{cases} \tag{6}$$

in which $\psi^0 = \frac{1}{2}{A^0}_{klmn}\epsilon_{kl}\epsilon_{mn} + \frac{1}{2}{B^0}_{klmn}e_{kl}e_{mn} + \frac{1}{2}{C^0}_{klmnpq}\gamma_{klm}\gamma_{npq} + {E^0}_{klmn}\epsilon_{kl}e_{mn}$, where ${A^0}_{klmn}$, ${B^0}_{klmn}$, ${C^0}_{klmnpq}$, ${E^0}_{klmn}$ are the initial constitutive moduli, and E is the material Young's modulus for the classical medium at the microscale used to construct an equivalent micromorphic continuum through the homogenization technique [15].

3 Discontinuous failure

In this study, it is proposed an extension of discontinuous failure concepts, well established for classical medium [9, 10, 21–24], to elastic-degrading micromorphic media. Based on the non-local formulation of the micromorphic theory, the localization analysis is constructed from a macroscale perspective and a microscale one, which are introduced in the next sections.

3.1 Macroscale discontinuity

For the purpose of this study a discontinuity in a micromorphic medium is a singular surface on which the second spatial and second time derivatives of $\boldsymbol{u}$ and $\boldsymbol{\phi}$ have jumps, while $\boldsymbol{u}$ and $\boldsymbol{\phi}$ together with their first derivatives are continuous [11, 25]. Denoting a jump of a variable a across the wavefront by double brackets ($[\![a]\!] = a^+ - a^-$) and $\boldsymbol{n}$ a unit vector normal to the singular surface, and applying Maxwell's theorem [11] to the continuous fields, the following conditions hold

$$[\![\dot{u}_{i,j}]\!] = a_i n_j, \quad [\![\dot{\phi}_{ij,k}]\!] = A_{ij} n_k \tag{7}$$

wherein the overdot stands for the time derivative, and a_i and A_{ij} are the vectorial and second-order tensorial amplitudes of the jumps. Hence, from the macroscale perspective, taking into account eq. (2), only the strain tensors ϵ_{ij} and γ_{ijk} have jumps while e_{ij} remains continuous. Then, considering a damaged state and a discontinuity only in the displacement u_i ($[\![\ddot{u}_i]\!] \neq 0$ and $[\![\ddot{\phi}_{ij}]\!] = 0$), the following condition can be written based on eq. (5)

$$Q^{t,aa}_{lp} a_p = 0\,, \quad Q^{t,aa}_{lp} = (1-D)A^0_{klpq} n_q n_k - \left(\frac{\partial \mathscr{D}(\Gamma_{eq})}{\partial \Gamma_{eq}}\right) t^0_{kl} n_k \left(\frac{\partial \Gamma_{eq}}{\partial \epsilon_{pq}}\right) n_q\,. \tag{8}$$

where the macroscale tangent acoustic tensor $Q^{t,aa}_{lp}$ has been introduced and $t^0_{kl} = A^0_{klmn}\epsilon_{mn} + E^0_{klmn}e_{mn}$. Therefore, localization is detected when eq. (8) is satisfied for a normal direction $\boldsymbol{n}$, i.e., when $\det(Q^{t,aa}_{lp}) = 0$.

3.2 Microscale discontinuity

Similarly, the condition for the onset of localization from a microscale perspective can be determined considering now the hypothesis of a discontinuity related only to the strain field e_{ij}. Therefore,

$$[\![\dot{\epsilon}_{ij}]\!] = 0\,, \quad [\![\dot{e}_{ij}]\!] = b_i n_j\,, \quad \text{and} \quad [\![\dot{\gamma}_{ijk}]\!] = 0 \tag{9a,b,c}$$

where b_i is the vectorial amplitude of the jump and n_j is a unit vector normal to the singular surface. Regarding a damaged state, the microscale localization condition is defined

$$det(Q^{t,b}_{lp}) = 0\,, \quad Q^{t,b}_{lp} = (1-D)B^0_{klpq} n_q n_k - \left(\frac{\partial \mathscr{D}(\Gamma_{eq})}{\partial \Gamma_{eq}}\right) s^0_{kl} n_k \left(\frac{\partial \Gamma_{eq}}{\partial e_{pq}}\right) n_q\,. \tag{10}$$

where $Q^{t,b}_{lp}$ is the microscale tangent acoustic tensor and $s^0_{kl} = E^0_{mnkl}\epsilon_{mn} + B^0_{klmn}e_{mn}$.

4 Localization analysis: uniaxial stress state

The behavior of the macroscale and microscale micromorphic acoustic tensors is here investigated for a uniaxial stress state, as illustrated in Fig. 1. The Finite Element Method (FEM) simulation has been performed with the open-source software INSANE (INteractive Structural ANalysis Environment), which source code is freely available at the Git repository `https://git.insane.dees.ufmg.br/insane/insane.git`. The 3×1 m^2 panel is discretized with three 4-node quadrilateral elements with four integration points and subjected to a traction load q. The model is simulated considering both classical and micromorphic theories.

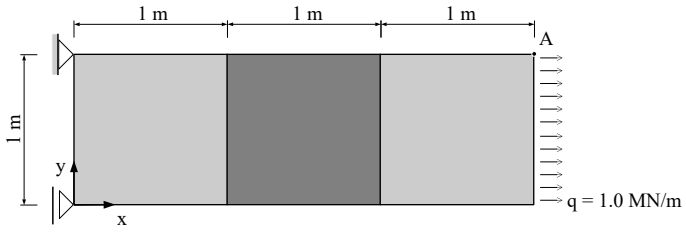


Figure 1. Uniaxial stress state: geometry

For the classical analysis, the constitutive models proposed by Simo-Ju [17], Ju [18], Marigo [19], and Mazars-Lemaitre [20] are employed associated to a Young's modulus $E = 20000$ MPa and a Poisson's ratio $\nu = 0.2$. For the micromorphic continuum, the generalized models presented in eq. (6) are used considering the same Young's modulus ($E = 20000$ MPa) and Poisson's ratio ($\nu = 0.2$). The adoption of the classical constants for both analysis is made possible by a homogenization technique [15] which considered, for this example, a square microcontinuum with 0.5 m to obtain the micromorphic constitutive equations.

The appropriated parameters for each constitutive model were adopted considering an exponential damage law described by

$$\mathscr{D}(\Gamma_{eq}) = 1 - \frac{K_0}{\Gamma_{eq}} \left(1 - \alpha + \alpha e^{-\beta(\Gamma_{eq} - K_0)}\right) \tag{11}$$

in which K_0 is the threshold for the equivalent strain, α and β define the maximum damage level allowed and the damage evolution intensity, respectively, and Γ_{eq} is the generalized equivalent strain (see eq. (6)). The central element, highlighted in Fig. 1, was considered to be less resistant and associated to material M2 while the two other elements were associated to material M1. The parameters adopted for M1 and M2 considered in the classical and the micromorphic analysis are detailed in Table 1 for each damage model.

Table 1. Uniaxial stress state: constitutive parameters

Constitutive model	Parameters
Simo-Ju	M1: $\alpha = 0.999$; $\beta = 30.0$; $K_0 = 0.0148$
	M2: $\alpha = 0.999$; $\beta = 30.0$; $K_0 = 0.0133$
Ju	M1: $\alpha = 0.999$; $\beta = 15.0$; $K_0 = 0.00011$
	M2: $\alpha = 0.999$; $\beta = 15.0$; $K_0 = 0.000099$
Marigo	M1: $\alpha = 0.99$; $\beta = 4000.0$; $K_0 = 0.0001049$
	M2: $\alpha = 0.99$; $\beta = 4000.0$; $K_0 = 0.000094$
Mazars-Lemaitre	M1: $\alpha = 0.99$; $\beta = 4000.0$; $K_0 = 0.0001049$
	M2: $\alpha = 0.99$; $\beta = 4000.0$; $K_0 = 0.000094$

The loading process is driven by the displacement control method assuming an increment of 1×10^{-5} m for the horizontal displacement of node A (see Fig. 1), and a tolerance for the convergence of 10^{-4} in displacements.

Figure 2 presents the results for the analysis wherein the relation between the horizontal displacement for node A (Fig. 1) and the load factor q is given. Steps highlighted with an mark "X" indicate the first detection of strain localization.

The micromorphic continuum, as it can be seen in the obtained equilibrium paths illustrated in Fig. 2(b), presents some differences when compared to the classical analysis. For the micromorphic analysis the peak load is higher and the post-peak branch is not directly correlated to the exponential damage law adopted.

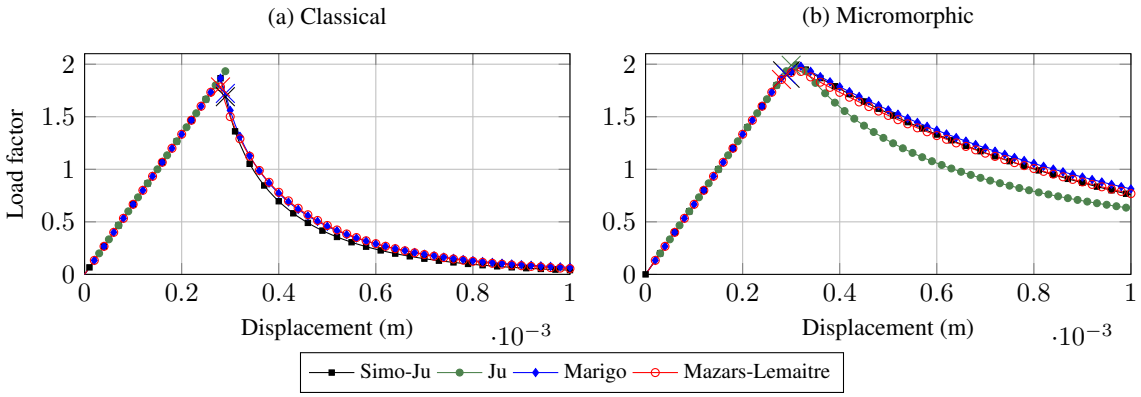


Figure 2. Uniaxial stress state: load factor versus horizontal displacement

In order to study the behavior of the acoustic tensor in a micromorphic medium, the minimum value of the acoustic tensor is found throughout the nonlinear process applying a numerical technique based on a Cartesian parametrization to determine the orientation $\boldsymbol{n}$ at which the classical and micromorphic acoustic tensors become singular [26, 27].

The results for the minimum value computed for the determinant of the considered acoustic tensor $\boldsymbol{Q}$ at each step of the nonlinear analysis for the four damage models considered is presented in Fig. 3. During elastic loading, the determinant remains constant for classical and micromorphic continuum. However, once damage initiates, the determinant in the classical and microscale analyses becomes negative, indicating localization. In contrast, for the macroscale micromorphic analysis, the determinant experiences a sharp decrease but remains positive throughout the loading process.

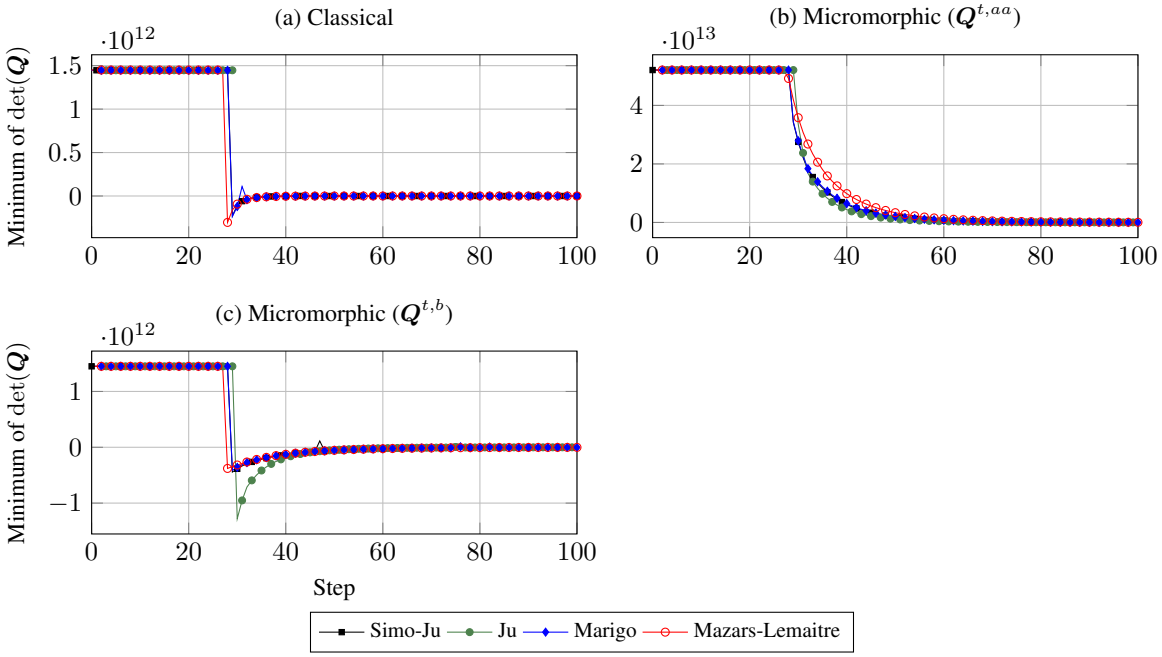
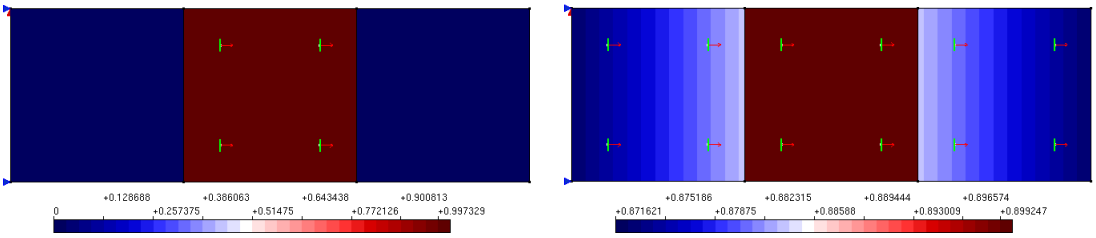


Figure 3. Uniaxial stress state: minimum of $\det(\boldsymbol{Q})$

Finally, the damage distribution for the final step of the nonlinear analysis (see Fig. 2) is shown in Fig. 4. For the classical continuum (Fig. 4(a)), material degradation is entirely confined to the weakened element, while the surrounding elements unload. Strain localization is observed at all material points within this element. The red line and arrow represent the numerically computed unit vector $\boldsymbol{n}$, while the green line qualitatively indicates the potential crack direction.

In contrast, the micromorphic continuum (Fig. 4(b)) provides a more accurate representation of localization where all three elements eventually exhibit damage as the analysis progresses. Strain localization is identified in all material points throughout the model adopting the microscale acoustic tensor $\boldsymbol{Q}^{t,b}$. Furthermore, the predicted direction for discontinuity insertion fits well with the expectations for a tensile problem.



(a) Classical continuum (b) Micromorphic continuum

Figure 4. Uniaxial stress state: damage (Mazars-Lemaitre damage model)

5 Conclusions

This work has investigated the definition of a reliable criterion for discontinuity initiation in the micromorphic continuum, enabling the transition from continuous damage to a discontinuous phase. Owing to its non-local formulation, the micromorphic continuum exhibits inherent regularization properties in the presence of strain localization.

An initial attempt to formulate a localization criterion at the macroscale, based solely on measures related to the macroscopic fields, involved the singularity of a macroscale acoustic tensor. However, this approach failed to indicate strain localization. To overcome this limitation, a microscale micromorphic acoustic tensor was proposed and validated through numerical simulations, successfully identifying strain localization and determining the discontinuity direction in generalized continua.

These results open perspectives for new continuous–discontinuous transition strategies for damage and fracture in generalized media, and contribute to a deeper understanding of strain localization and regularization mechanisms.

Acknowledgements. The authors gratefully acknowledge the support from the Brazilian research agencies CAPES (in portuguese: *Coordenação de Aperfeiçoamento de Pessoal de Nível Superior*), FAPEMIG (in portuguese: *Fundação de Amparo à Pesquisa do Estado de Minas Gerais*), and CNPq (in portuguese: *Conselho Nacional de Desenvolvimento Científico e Tecnológico*; grant 316240/2021-4).

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] F. Mukhtar and A. El-Tohfa. A review on fracture propagation in concrete: Models, methods, and benchmark tests. *Engineering Fracture Mechanics*, vol. 281, pp. 109100, 2023.

[2] M. Jirásek. Modeling of localized damage and fracture in quasibrittle materials. In P. A. Vermeer, H. J. Herrmann, S. Luding, W. Ehlers, S. Diebels, and E. Ramm, eds, *Continuous and Discontinuous Modelling of Cohesive-Frictional Materials*, pp. 17–29. Springer Berlin Heidelberg, Berlin, Heidelberg, 2001.

[3] M. Jirásek and T. Zimmermann. Embedded crack model. Part II: combination with smeared cracks. *International Journal for Numerical Methods in Engineering*, vol. 50, n. 6, pp. 1291–1305, 2001a.

[4] M. Jirásek and T. Zimmermann. Embedded crack model: I. basic formulation. *International Journal for Numerical Methods in Engineering*, vol. 50, n. 6, pp. 1269–1290, 2001b.

[5] J. Oliver, A. Huespe, M. Pulido, and E. Chaves. From continuum mechanics to fracture mechanics: the strong discontinuity approach. *Engineering Fracture Mechanics*, vol. 69, n. 2, pp. 113–136, 2002.

[6] C. Comi, S. Mariani, and U. Perego. An extended fe strategy for transition from continuum damage to mode i cohesive crack propagation. *International Journal for Numerical and Analytical Methods in Geomechanics*, vol. 31, n. 2, pp. 213–238, 2007.
[7] A. Simone, G. N. Wells, and L. J. Sluys. From continuous to discontinuous failure in a gradient-enhanced continuum damage model. *Computer Methods in Applied Mechanics and Engineering*, vol. 192, n. 41, pp. 4581–4607, 2003.
[8] T. Belytschko, H. Chen, J. Xu, and G. Zi. Dynamic crack propagation based on loss of hyperbolicity and a new discontinuous enrichment. *International Journal for Numerical Methods in Engineering*, vol. 58, n. 12, pp. 1873–1905, 2003.
[9] J. Hadamard. *Leçons sur la propagation des ondes et les équations de l'hydrodynamique*. Librairie Scientifique A. Hermann, Paris, 1903.
[10] R. Hill. A general theory of uniqueness and stability in elastic-plastic solids. *Journal of the Mechanics and Physics of Solids*, vol. 6, n. 3, pp. 236–249, 1958.
[11] V. A. Eremeyev, L. P. Lebedev, and M. J. Cloud. Acceleration waves in the nonlinear micromorphic continuum. *Mechanics Research Communications*, vol. 93, pp. 70–74, 2018.
[12] A. C. Eringen and E. S. Şuhubi. Nonlinear theory of simple micro-elastic solids – i. *International Journal of Engineering Science*, vol. 2, n. 2, pp. 189–203, 1964.
[13] R. D. Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis*, vol. 16, pp. 51–78. Communicated by R. A. Toupin, 1964.
[14] E. S. Suhubi and A. C. Eringen. Nonlinear theory of simple micro-elastic solids – II. *International Journal of Engineering Science*, vol. 2, n. 4, pp. 389–404, 1964.
[15] L. L. Silva, R. L. d. S. Pitangueira, and S. S. Penna. Multiscale numerical strategy for micromorphic description of quasi-brittle media from classical elastic damage models at the microscale. *Applied Mathematical Modelling*, vol. 109, pp. 52–76, 2022.
[16] P. D. Reges, R. L. Pitangueira, and L. L. Silva. Elastic degradation models for the micromorphic continuum. *International Journal of Non-Linear Mechanics*, pp. 104450, 2023.
[17] J. Simo and J. Ju. Strain- and stress-based continuum damage models–i. formulation. *International Journal of Solids and Structures*, vol. 23, n. 7, pp. 821 – 840, 1987.
[18] J. Ju. On energy-based coupled elastoplastic damage theories: Constitutive modeling and computational aspects. *Int. J. Solids Struct.*, vol. 25, n. 7, pp. 803–833, 1989.
[19] J. Marigo. Modelling of brittle and fatigue damage for elastic material by growth of microvoids. *Engineering Fracture Mechanics*, vol. 21, n. 4, pp. 861 – 874, 1985.
[20] J. Mazars and J. Lemaitre. *Application of Continuous Damage Mechanics to Strain and Fracture Behavior of Concrete*, pp. 507–520. Springer Netherlands, Dordrecht, 1985.
[21] J. W. Rudnicki and J. R. Rice. Conditions for the localization of deformation in pressure-sensitive dilatant materials. *Journal of the Mechanics and Physics of Solids*, vol. 23, n. 6, pp. 371–394, 1975.
[22] N. S. Ottosen and K. Runesson. Properties of discontinuous bifurcation solutions in elasto-plasticity. *International Journal of Solids and Structures*, vol. 27, n. 4, pp. 401–421, 1991.
[23] E. Rizzi, I. Carol, and K. Willam. Localization Analysis of Elastic Degradation with Application to Scalar Damage. *Journal of Engineering Mechanics*, vol. 121, n. 4, pp. 541–554. Publisher: American Society of Civil Engineers, 1995.
[24] M. Jirásek. Mathematical analysis of strain localization. *Revue Européenne de Génie Civil*, vol. 11, n. 7-8, pp. 977–991, 2007.
[25] C. Truesdell and W. Noll. The Non-Linear Field Theories of Mechanics. In C. Truesdell, W. Noll, and S. S. Antman, eds, *The Non-Linear Field Theories of Mechanics*, pp. 1–579. Springer, Berlin, Heidelberg, 2004.
[26] A. Mota, Q. Chen, J. Foulk III, J. Ostien, and Z. Lai. A Cartesian parametrization for the numerical analysis of material instability. *International Journal for Numerical Methods in Engineering*, vol. 108, n. 2, pp. 156–180. _eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/nme.5228, 2016.
[27] L. A. F. Fioresi, R. L. S. Pitangueira, and S. S. Penna. Numerical technique for strain localization analysis considering a Cartesian parameterization. *Journal of the Brazilian Society of Mechanical Sciences and Engineering*, vol. 42, n. 3, pp. 145, 2020.