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Differential Relations for Deflected Shape and Internal Forces in 2D Frames with Straight and Curved Elements

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Abstract. This paper presents the differential relations and the development of an algorithm that calculates the deflected shape of frames with both circular arcs and straight elements, subjected to various types of loads under linear-elastic conditions. The program is developed in Python and is based on the Finite Element Method. Differential equations describing displacements and rotations along the structure are presented. The stiffness matrices and load vectors are obtained by inverting the corresponding flexibility matrices, which are calculated using the Force Method. A Gauss-Legendre Quadrature is employed to evaluate some integrals. An example is provided, calculated and compared with results from commercial software, where curved bars must be discretized into multiple elements. The displacements and rotations calculated by the algorithm are shown to agree very well, and in most cases, are closer to that calculated with the Virtual Work Method, while requiring less demanding input data.

Keywords: Differential relations; Curved elements; Finite Element Method; Python; Gauss-Legendre Quadrature.

1 Introduction

Structural analysis plays a fundamental role in civil engineering, providing essential techniques and tools to understand how structures respond to different types of loads. According to Soriano [1], its main objective is to correlate the external forces applied to the system with the resulting displacements, support reactions, and internal stresses, allowing for an accurate assessment of structural performance under various conditions. Several Finite Element Method (FEM) based software, such as Ftool, are widely used for the structural analysis of regular 2D frames. The present work aims to develop an additional component for PAE (*Programa de Análise de Estruturas*), a similar program also based on the FEM, developed from works such as Coimbra et al. [2], that implemented an articulation that allows free translational movement into the program, and Melo et al. [3], that introduced thermal loading analysis for straight and curved elements. It is designed for the analysis of non-conventional structures, such as those with variable inertias and arches, in addition to being capable of handling different types of loads, and includes versions and modules written in Fortran, Java and Python, which have been continuously improved through various studies.

Lima and Soares [4] proposed an additional module for the study of grids with straight and curved bars, whereas Rego et al. [5] applied Gauss quadrature to the evaluation of these structures, which increased the accuracy of the results, especially in more complex configurations. Morais and Soares [6] developed an interpolation function for finite elements in circular arcs, allowing for more accurate modeling of these elements. Furthermore, Oliveira et al. [7] contributed to the development of an object-oriented graphical interface, making the program more accessible and intuitive for users.

The proposed module aims to enable the deflected shape analysis of frame structures composed of straight and circular arc elements, using differential equations that relate internal forces to rotations and both horizontal

and vertical displacements. The methodology adopted follows an approach similar to that used for conventional straight bars with constant cross-section, under linear-elastic conditions. The code was developed in Python, chosen for its modern features and accessibility.

2 Methodology

Initially, the code was designed for the analysis of structures composed solely of straight bars, for which stiffness matrices and load vectors are widely known and well documented, as exemplified in the work of Martha [8]. For curved elements, the stiffness matrices and load vectors were obtained by inverting the corresponding flexibility matrices, calculated using the Force Method. To ensure the accuracy of the integrals involved in this process, the Gauss-Legendre numerical method was implemented, which is widely recognized for its ability to compute integrals with high precision.

2.1 Straight Elements

To derive the differential equations, an analysis was carried out based on established formulations for the deflected shape in straight elements, which relate rotations and displacements to internal forces. Martha [8] infers that the behavior of beams under bending was formalized in the early 19th century by Navier, who developed a theory that, by neglecting shear deformations, establishes a differential equation relating the transverse displacements of a beam with constant moment of inertia to the rate of transversely distributed loading. Such theory is based on several fundamental assumptions, namely:

a. Displacements are small compared to the dimensions of the cross-section;
b. Bernoulli's hypothesis, which states that plane cross-sections of a beam remain plane and perpendicular to the deformed beam's neutral axis after bending deformation;
c. The material follows a linear-elastic behavior (Hooke's Law).

Thus, the fourth-order differential equation that relates internal forces and deflected shape for straight bars is given by eq. (1):

$$\frac{d^4v}{dx^4} = \frac{q(x)}{E \cdot I} \implies \frac{d^3v}{dx^3} = \frac{C(x)}{E \cdot I} \implies \frac{d^2v}{dx^2} = \frac{M(x)}{E \cdot I} \implies \frac{dv}{dx} = \theta(x) \implies v(x) \tag{1}$$

Where $q(x)$ is the distributed load, $C(x)$ is the shear force, $M(x)$ is the bending moment, $\theta(x)$ is the cross-sectional rotation due to bending, $v(x)$ is the beam's transverse displacement, E is the material's modulus of elasticity, and I is the cross-sectional moment of inertia.

In order to calculate the rotation and deflection of the beam, the bending moment equation is integrated twice. This process introduces constants, which are determined using boundary conditions.

2.2 Circular arc elements

For curved elements whose axis follow a circular arc, a methodology similar to that used for straight bars was applied, with specific adaptations. Figure 1 shows the finite element used in PAE for the consideration of circular arcs, where α_1 and α_2 are the initial and final angles measured clockwise from the *x*-axis, while R represents the radius of the arc and α stands for the reference angle. The element has two nodes with three degrees of freedom (DOF) each. The parameters u, v and θ represent the horizontal, vertical, and rotational DOF, respectively, while the subscripts i and j denote the initial and final nodes.

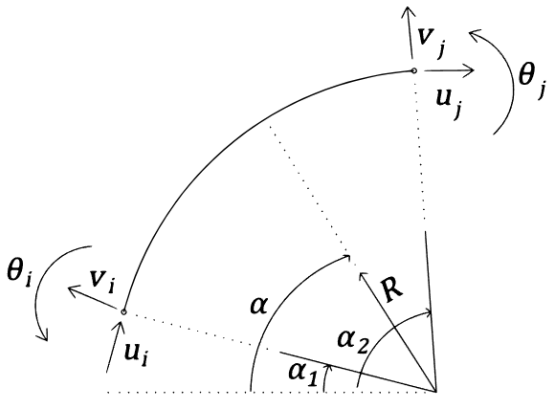


Figure 1. Generic circular arc element

From the bending moment equation, the necessary integrations are carried out to calculate the deformation along the element, neglecting the contributions of shear and axial forces (the relationship between these neglected internal forces and the distributed load in arches can be found in Soriano [1]). The resulting differential relations are presented in eq. (2) for the cross-sectional rotation, and eqs. (3) and (4) for the vertical and horizontal displacements, respectively, both expressed in the global coordinate system.

$$\frac{1}{R} \cdot \frac{d\theta(\alpha)}{d\alpha} = M(\alpha) \tag{2}$$

$$\frac{1}{R} \cdot \frac{dv(\alpha)}{d\alpha} = \theta(\alpha) \cdot sin(\alpha) \tag{3}$$

$$\frac{1}{R} \cdot \frac{du(\alpha)}{d\alpha} = -\theta(\alpha) \cdot cos(\alpha) \tag{4}$$

Boundary conditions are applied based on the structures' supports in order to determine the integration constants from the governing integrals. This leads to equations that represent the deflected shape of the element and allow for the calculation of displacements in both directions, as well as rotation, at any given point along the structure.

2.3 General equation

The equations aforementioned, along with some results from the algorithm, were used to develop general expressions that could be computationally implemented to generate the deflected shape and rotational behavior of any circular arc finite element. The parameters used include: concentrated loads in the horizontal and vertical directions (denoted F and P, respectively); the bending moment at the initial node (M_i); horizontal and vertical distributed loads (q_x and q_y); the initial angle (α_1); Young's modulus (E_c); the moment of inertia (I); and the DOFs at the initial node (u_i, v_i, and θ_i). The equation that describes the vertical displacements, for example, is presented in eq. 5.

$$v(\alpha) = \Big[E_c{\cdot}I_f{\cdot}R{\cdot}\,\theta_i \cdot cos(\alpha_1) + E_c{\cdot}I_f{\cdot}v_i - F{\cdot}R^3{\cdot}\frac{sin(\alpha_1)^2}{2} - F{\cdot}R^3 + M_i{\cdot}R^2 \cdot sin(\alpha_1) + \frac{P{\cdot}R^3{\cdot}\alpha_1}{2} - \frac{P{\cdot}R^3{\cdot}sin(2\alpha_1)}{4} - \frac{R^4{\cdot}\alpha_1{\cdot}q_y{\cdot}cos\,(\alpha_1)}{2} + \frac{R^4{\cdot}q_x{\cdot}sin(\alpha_1)^3}{6} + R^4 \cdot q_x \cdot sin(\alpha_1) - \frac{R^4{\cdot}q_y{\cdot}sin(\alpha_1)^3}{6} + \frac{R^4{\cdot}q_y{\cdot}sin(\alpha_1)}{2} - R \cdot (12 \cdot E_c{\cdot}I_f{\cdot}\theta_i \cdot cos(\alpha) + 12 \cdot F{\cdot}R^2 \cdot \alpha \cdot sin(\alpha_1) \cdot cos(\alpha) - 12 \cdot F{\cdot}R^2 \cdot \alpha_1 \cdot sin(\alpha_1) \cdot cos(\alpha) - 6 \cdot F{\cdot}R^2 \cdot \alpha_1 \cdot sin(\alpha)^2 - 12 \cdot F{\cdot}R^2 \cdot cos(\alpha - \alpha_1) - 12 \cdot M_i{\cdot}R{\cdot}\alpha \cdot cos(\alpha) + 12 \cdot M_i{\cdot}R{\cdot}\alpha_1 \cdot cos(\alpha) + 12 \cdot M_i{\cdot}R{\cdot}sin(\alpha) + 12 \cdot P{\cdot}R^2 \cdot \alpha \cdot cos(\alpha) \cdot cos(\alpha_1) + 6 \cdot \tag{5}$$

$$P{\cdot}R^2 \cdot \alpha - 12 \cdot P{\cdot}R^2 \cdot \alpha_1 \cdot cos(\alpha) \cdot cos(\alpha_1) - 6 \cdot P{\cdot}R^2 \cdot sin(\alpha) \cdot cos(\alpha) - 12 \cdot P{\cdot}R^2 \cdot sin(\alpha - \alpha_1) - 6{\cdot}R^3 \cdot \alpha \cdot q_x \cdot sin(\alpha_1)^2 \cdot cos(\alpha) - 3 \cdot R^3 \cdot \alpha \cdot q_x \cdot cos(\alpha) + 6{\cdot}R^3 \cdot \alpha \cdot q_y \cdot sin(\alpha_1)^2 \cdot cos(\alpha) - 9{\cdot}R^3 \cdot \alpha \cdot q_y \cdot cos(\alpha) - 6{\cdot}R^3 \cdot \alpha \cdot q_y \cdot cos(\alpha_1) + 6{\cdot}R^3 \cdot \alpha_1 \cdot q_x \cdot sin(\alpha_1)^2 \cdot cos(\alpha) + 3{\cdot}R^3 \cdot \alpha_1 \cdot q_x \cdot cos(\alpha) - 6{\cdot}R^3 \cdot \alpha_1 \cdot q_y \cdot sin(\alpha_1)^2 \cdot cos(\alpha) + 9{\cdot}R^3 \cdot \alpha_1 \cdot q_y \cdot cos(\alpha) - R^3 \cdot q_x \cdot sin(\alpha)^3 + 6{\cdot}R^3 \cdot q_x \cdot sin(\alpha)^2 \cdot sin(\alpha_1) + 6{\cdot}R^3 \cdot q_x \cdot sin(\alpha_1)^2 \cdot sin(\alpha) + 3 \cdot R^3 \cdot q_x \cdot sin(\alpha) + 9{\cdot}R^3 \cdot q_x \cdot sin(\alpha_1) \cdot cos(\alpha) \cdot cos(\alpha_1) + R^3 \cdot q_y \cdot sin(\alpha)^3 - 6{\cdot}R^3 \cdot q_y \cdot sin(\alpha_1)^2 \cdot sin(\alpha) + 6 \cdot R^3 \cdot q_y \cdot sin(\alpha) \cdot cos(\alpha) \cdot cos(\alpha_1) + 9{\cdot}R^3 \cdot q_y \cdot sin(\alpha) - 9{\cdot}R^3 \cdot q_y \cdot sin(\alpha_1) \cdot cos(\alpha) \cdot cos(\alpha_1) \big)/12\Big] / (E_c{\cdot}I_f)$$

3 Results

Figure 2 presents a 2D frame selected as an example, composed of two straight elements and one circular arc, subjected to both concentrated and uniformly distributed loads. In the input data for the straight elements, it is essential to define the coordinates of the nodes, whereas the curved element, described in polar coordinates, requires only the initial and final angles, as well as its radius. It was considered for the structure an $E_c = 20GPa$ and $I_f = 1.067 \cdot 10^{-3}$, which are the Young's Modulus and the moment of inertia, respectively.

Table 1 shows the results obtained for the proposed structure, comparing the solutions of the differential equations implemented in PAE with those generated by the Ftool software. For comparison purposes, the structure was modeled in Ftool with the arc discretized as 36 straight bar elements.

Despite the reasonable agreement between the results obtained by both methods, PAE stands out by significantly simplifying the structural analysis process. This advantage stems from its dedicated element for curved structures, while Ftool relies solely on straight elements. Additionally, the implementation of general equations in PAE provides results continuously along the entire element, including intermediate points between nodes, eliminating the need to subdivide the structure into multiple smaller segments—making the procedure more efficient, direct, and intuitive.

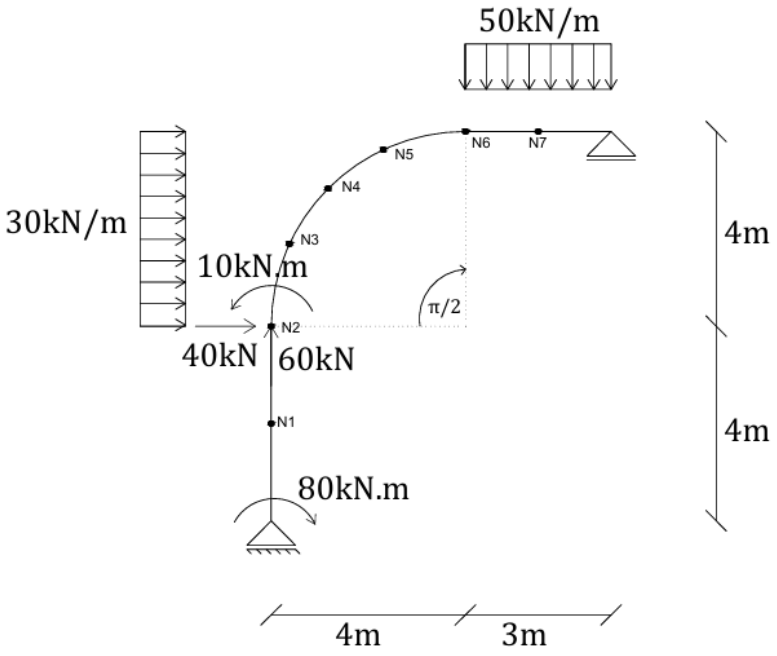


Figure 2. Frame structure used for validation

Table 1. Results

Node	Method	x (m)	y (m)	Displacement Dx (m)	Displacement Dy (m)	Rotation (rad)
1	General equation	0,000	−2,000	$5{,}117 \cdot 10^{-1}$	$1{,}523 \cdot 10^{-7}$	$-2{,}421 \cdot 10^{-1}$
	Ftool			$5.218 \cdot 10^{-1}$	$1{,}725 \cdot 10^{-12}$	$-2{,}472 \cdot 10^{-1}$
2	General equation	0,000	0,000	$9{,}485 \cdot 10^{-1}$	$3{,}436 \cdot 10^{-16}$	$-1{,}896 \cdot 10^{-1}$
	Ftool			$9{,}686 \cdot 10^{-1}$	$3{,}450 \cdot 10^{-12}$	$-1{,}946 \cdot 10^{-1}$
3	General equation	0,375	1,690	1,217	$-5{,}536 \cdot 10^{-2}$	$-1{,}254 \cdot 10^{-1}$
	Ftool			1,245	$-5{,}738 \cdot 10^{-2}$	$-1{,}301 \cdot 10^{-1}$
4	General equation	1,172	2,828	1,333	$-1{,}318 \cdot 10^{-1}$	$-7{,}121 \cdot 10^{-2}$
	Ftool			1,363	$-1{,}372 \cdot 10^{-1}$	$-7{,}473 \cdot 10^{-2}$
5	General equation	2,310	3,625	1,368	$-1{,}826 \cdot 10^{-1}$	$-2{,}126 \cdot 10^{-2}$
	Ftool			1,403	$-1{,}909 \cdot 10^{-1}$	$-2{,}260 \cdot 10^{-2}$
6	General equation	4,000	4,000	1,369	$-1{,}727 \cdot 10^{-1}$	$2{,}981 \cdot 10^{-2}$
	Ftool			1,404	$-1{,}808 \cdot 10^{-1}$	$3{,}122 \cdot 10^{-2}$
7	General equation	5,500	4,000	1,369	$-1{,}029 \cdot 10^{-1}$	$6{,}070 \cdot 10^{-1}$
	Ftool			1,404	$-1{,}077 \cdot 10^{-1}$	$6{,}357 \cdot 10^{-2}$

4 Conclusions

The proposed module, integrated into PAE, performs linear-elastic analysis of frame structures composed of conventional elements – both straight bars and circular arcs – subjected to nodal and distributed loads. The tool provides, in a practical manner, the displacements of a complex structure along its entire domain through a simple and objective data input process. This advantage over similar, well-established software makes PAE a valuable complementary tool both in teaching and learning contexts, as well as in research within related fields.

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