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A comparative study of edge and cell-based smoothing domains in SPIM-FEM coupling for phase-field modeling of hydraulic fracture

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Abstract. Hydraulic fracturing is a multi-physical process characterized by the occurrence of complex crack patterns, accentuated by the presence of the fluid pressure load on the crack surface. For this reason, the choice of a suitable model for treating the cracking process and the non-linear behavior of the material is very important. An interesting approach is the use of phase-field model (PFM), which treats the onset and propagation of any number of cracks automatically, as part of the solution to a variational problem of energy minimization. The PFM has already been studied by several authors in the modeling of hydraulic fracturing and will be used in this work together with a coupling strategy of a meshfree method with the standard finite element method (FEM). The meshfree method adopted is the Smoothed Point Interpolation Method (SPIM). SPIM-FEM coupling is performed according to predetermined division of the problem domain into FEM and meshfree parts, seeking to ensure that the inelastic process occurs in the SPIM region. Edge- and cell-based smoothing domains are adopted, and their results will be compared through numerical simulations, which also validate and exemplify the proposed strategy. The comparison between the different strategies will consider the crack trajectory, the agreement of crack growth with results reported in the literature, and computational efficiency, measured by the number of iterations required for convergence. The computational implementation was carried out in INSANE, free software developed by the Department of Structural Engineering at the Federal University of Minas Gerais (UFMG).

Keywords: Hydraulic fracturing, Phase-field, Smoothed point interpolation methods, Coupled model

1 Introduction

Interest in understanding the behavior of materials, especially during failure and rupture processes, is not a recent topic, and therefore different models have been proposed and studied over the years. An interesting approach that has been explored by several authors, especially in the last 20 years, is phase field modeling for fracture. As highlighted by Wu et al. [1], the model, which allows the displacement field and any number of crack patterns to be obtained from the solution of an energy minimization problem, has as its main advantages the automatic determination of cracks, without the use of additional resources, allowing the representation of complex crack patterns, and can easily incorporate multiphysics problem formulations. In this sense, the application of phase-field models in the study of pressurized fracture problems is of interest to this work.

In pressurized fracture problems, the presence of fluid pressure along the crack faces often generates complex behavior such as rapid crack propagation, accompanied by bifurcation, nucleation, and abrupt changes in direction. In this sense, the phase-field model proves to be an important strategy in modeling this type of problem. The first work to record such an application is the one by Bourdin et al. [2], which will be adopted in this work.

Traditionally, the phase-field model is discretized using the finite element method (FEM), which yields solid results. However, other forms of numerical discretization can be employed. In this regard, the work by Novelli et al. [3] can be cited, who first proposed the study of the phase-field model for brittle fracture using a meshfree method called Smoothed Point Interpolation Methods (SPIMs). The fundamental characteristic of the SPIMs family is the construction of subdomains, called smoothing domains, from triangular background cells, which serve as the basis for constructing the deformation field. The paper presented consistent and promising results, which led to its extension to pressurized fracture problems by Ferreira et al. [4]. In the latter, the coupling strategy is employed so that the use of SPIM is limited to regions with phase-field propagation, with the rest of the domain

being discretized by FEM, which seeks to reduce the cost of preprocessing tasks intrinsic to the SPIM, such as the construction of shape functions. This work by Ferreira et al. [4] serves as the basis for the present paper, which seeks to compare the results obtained with the use of edge and cell-based smoothing domains in applications with pressurized fractures.

All implementations of this work were developed in the INSANE system, a segmented and expandable software developed at the Department of Structural Engineering at the Federal University of Minas Gerais. It is important to note that INSANE is responsible for pre-processing the model and performing the FEM-SPIM coupling based on the coordinates of the polygon that defines the region to be replaced by SPIM.

2 Phase-field modeling of hydraulic fracture

In phase-field models (PFM), localized fracture as shown in Fig. 1(a) is recast in a diffuse manner as shown in Fig. 1(b), represented by the phase-field variable ϕ ($0 \leq \phi \leq 1$). The width of the smooth degraded region $\mathcal{B}$ is determined by the length scale, l_0, an important material parameter in PFM. This work employs PFM in the study of fractures with hydraulic pressure load stimulation, following the formulation proposed by Bourdin et al. [2]. In this model, the pressure load appears on the crack surface (see Fig. 1(c)) and depends on the presence of phase-field gradient ($\nabla\phi$).

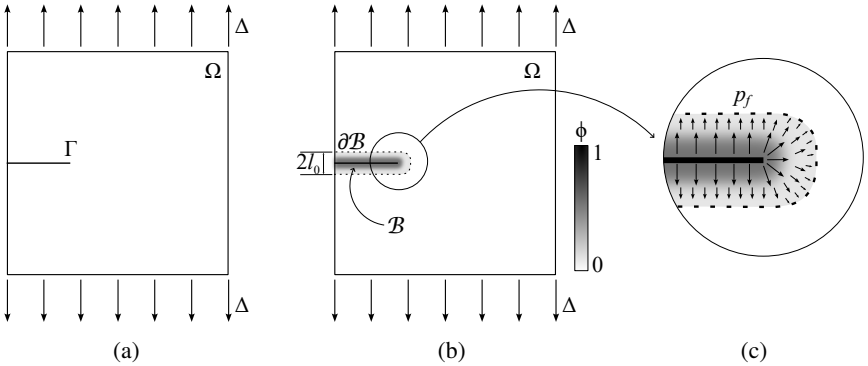


Figure 1. Sharp and diffusive crack modeling: (a) Sharp crack Γ; (b) Phase-field crack; (c) Pressure load p_f in the crack domain.

Considering the illustrated setting, the total energy functional of the phase-field model for pressurized fractures is given by:

$$E_t(\boldsymbol{u},\phi) = \int_\Omega \psi(\boldsymbol{\varepsilon}(\boldsymbol{u}),\phi)dV + \int_\mathcal{B} G_c\gamma(\phi,\nabla\phi)dV - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{u}dV - \int_{\partial\Omega_N} \boldsymbol{t}\cdot\boldsymbol{u}dA + \int_\Omega p_f\boldsymbol{u}\cdot\nabla\phi dV, \quad (1)$$

where ψ is the strain energy density, G_c is the critical energy release rate, γ is the crack surface density, and $\boldsymbol{b}$ and $\boldsymbol{t}$ are the body and surfaces forces, respectively.

One approach to obtaining the displacement u and phase-field ϕ solutions of eq. (1) is through alternating minimization with respect to each of these fields. Based on the weak form of the model equations, it is possible to develop the discretization from the chosen numerical model. The classic discretization based on the finite element method is widely adopted and produces consistent results in a number of applications, as presented by Wu et al. [1]. The present work, however, employs a coupling of finite element discretization with the Smoothed Point Interpolation Methods. The coupling between the two methods is interesting because it allows the crack region to be discretized using the meshless method, without the need to construct smoothing domains and assemble shape functions for the entire domain, which could increase the computational cost of the analysis. Furthermore, the coupling is straightforward, since both strategies have the Kronecker delta property.

2.1 SPIM discretization

The smoothing operation for the phase-field model for brittle fractures was used for the first time by Novelli et al. [3] and its application was extended to the phase-field model for pressurised fractures by Ferreira et al. [4]. Important aspects of this methodology will be outlined in the following.

The Smoothed Point Interpolation Methods (SPIMs) are a family of meshfree methods studied by Liu and Zhang [5]. In this strategy, the problem domain is divided into smoothing domains, which are regions where the strain field is considered constant, and they are constructed based on background triangular cells. In this paper, the background cells are the triangular elements of the initial finite element mesh. For plane problems, the smoothing domains can be constructed based on the node, on the cell or on the edge, the latter two approaches being used in this work. In cell-based, the integration cells are the background triangular cells themselves, as shown in Fig. 2(b), while in edge-based, the smoothed domains are constructed from each edge of the background triangular cells, as shown in Fig.2(c).

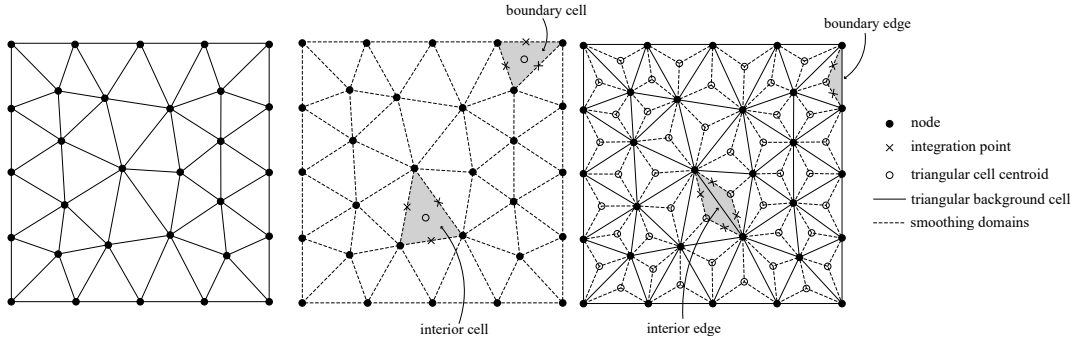


Figure 2. Smoothing domains: (a) Triangular background cells; (b) Cell-based; (c) Edge-based.

The SPIM shape functions are constructed over the course of the analysis and depend on the selection of support nodes. The background triangular cells are also used to select the support nodes, according to the so-called T-schemes. When the integration point is located on the edge of the cell, the T4-scheme is used, and the nodes of the two cells that have this edge in common are selected. This scheme is illustrated in Fig. 3(a). When the integration point is inside the cell, T3 or T6/3-schemes can be used. The T3-scheme selects the three nodes of the home cell as support nodes, as shown in Fig. 3(b), while the T6/3-scheme selects three more nodes, one from each neighbouring cell, except for cells on the edge of the domain, as shown in Fig. 3(c). This work uses the cell-based strategy with T4-scheme and edge-based with T3 and T6/3-schemes.

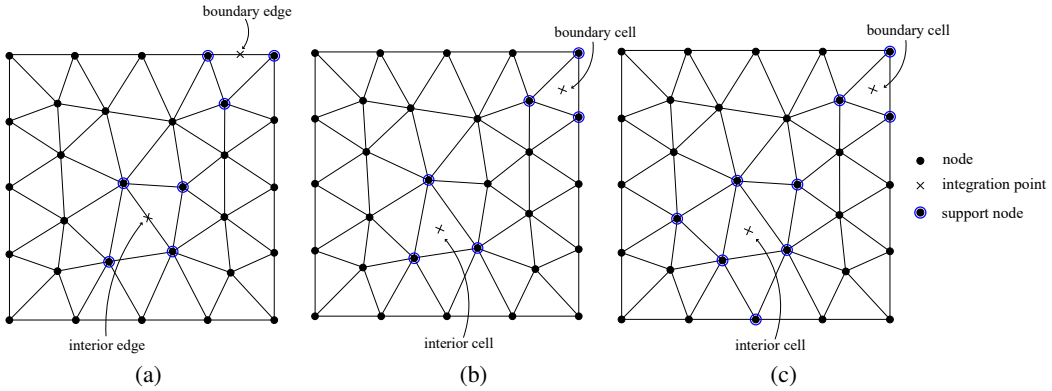


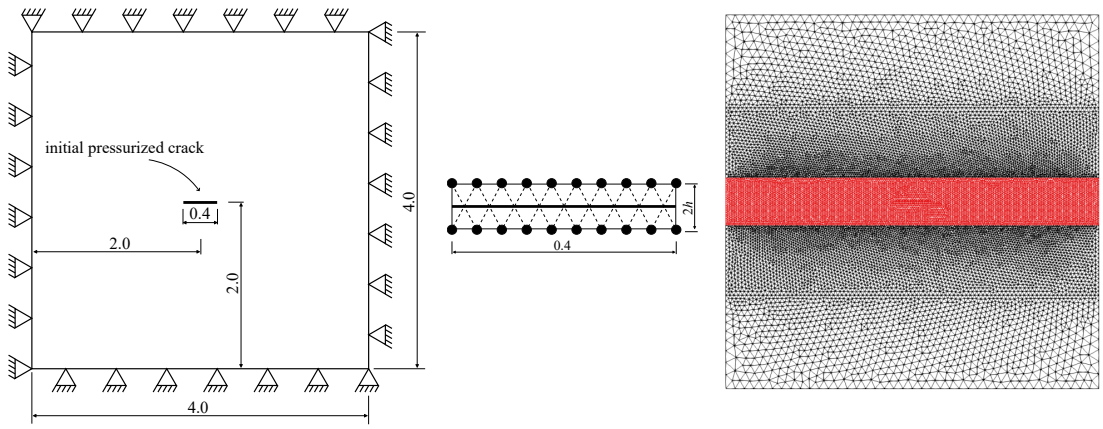
Figure 3. T-schemes for selecting support nodes: (a) T4-scheme; (b) T3-scheme; (c) T6/3-scheme.

3 Numerical results

This section will present some numerical results that validate the approach presented, with SPIM-FEM coupling in the discretization of the phase-field model for pressurized fractures, and show the application of the model. The region to be discretized by SPIM is defined at the beginning of the analysis, and the coupling is performed in the preprocessing stage. All simulations have an initial mesh with $h = l_0/2$ in the crack region and are simulated under plane strain conditions.

3.1 Single pressurized fracture

The first example was studied by Wheeler et al. [6] and involves a model with a pressurized internal crack, without external loading. The dimensions and boundary conditions are presented in Fig. 4(a). The initial crack is defined by applying $\phi = 1$ in a region of dimension $2h$, as shown in Fig. 4(b), where h is the characteristic size of the mesh in the crack region. The region highlighted in red in Fig. 4(c) is discretized by SPIM, while the rest of the domain is discretized by FEM. The dimensionless material parameters of the domain are: Young's modulus $E = 1.0$, Poisson's ration $\nu = 0.2$, critical energy release rate $G_c = 1.0$, and length scale $l_0 = 0.044$. The analysis consists of 30 steps in which the pressure load is increased linearly starting from $p_{f,0} = 0.1$.



(a) (b) (c)

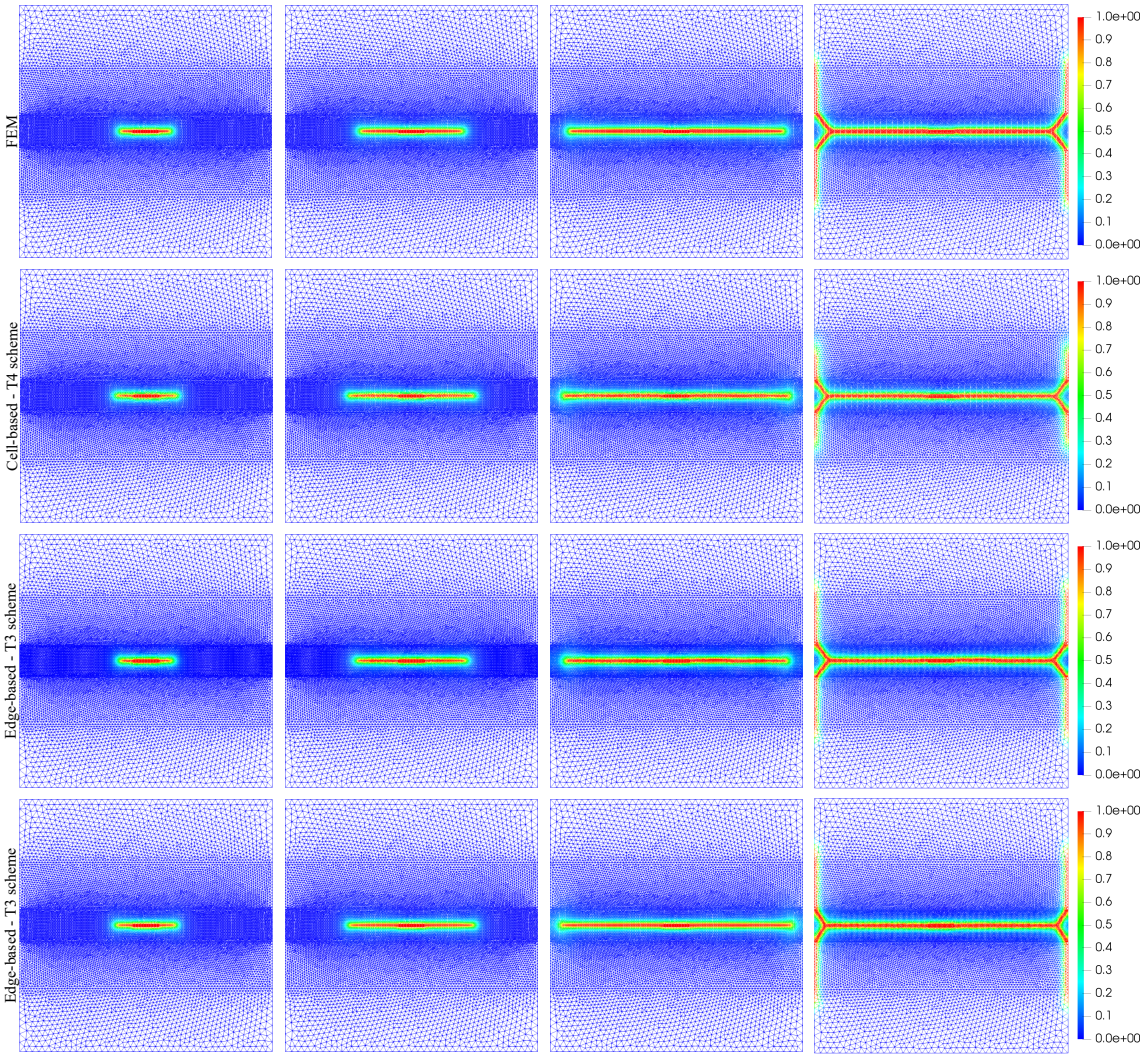
Figure 4. Single propagating fracture: (a) Problem setting; (b) Dimensions of the initial fracture; (c) Mesh with domain division in SPIM (in red) and FEM (in black).

The crack configuration for different steps of the analysis is shown in Fig. 5. The contour plots present a good agreement between the model discretized entirely with finite elements and those that employ the SPIM-FEM coupling presented in this work. No significant difference can be observed between the different smoothing domains and support node selection schemes. Figure 6 shows a comparison of the length of the pressurized crack for each step of the analysis, considering the different strategies studied. It can be seen that the SPIM-FEM coupling produced good results, consistent with those reported by Wheeler et al. [6].

3.2 Pressurized crack propagation in a heterogeneous quartzite-granite rock

The second example deals with a bi-material structure consisting of quartzite and granite, originally studied by Nguyen et al. [7]. In addition to the self-weight of the materials present in the original model, the effect of pressure load on the crack was added. The interface between the materials is inclined, as shown in Fig. 7(a), and the model has an initial crack at the top. The parameters used are (subscript q for quartzite and g for granite): Young's modulus $E_q = 41.1$ GPa and $E_g = 4.11$ GPa, Poisson's ratio $\nu_q = \nu_g = 0.25$, densities $\rho_q = \rho_g = 2690$ kg/m^3, length scale $l_{0,q} = l_{0,g} = 0.625$ m, critical energy release rate $G_{c,q} = 10000$ N/m and $G_{c,g} = 1000$ N/m. The region from the initial crack to the interface between the materials is discretized by SPIM (in red in Fig. 7(b)) while the rest of the domain is discretized by FEM (in black in Fig. 7(b)). This region was chosen due to the large difference between the properties of the materials, which indicates that this is the preferred path of the crack.

The loads included in the model are the self-gravitational load, highlighted by Nguyen et al. [7] as relevant in this type of geological application, and the pressure load $p_f = 1000$ N/m², which was added to the model and incremented throughout the analysis. The phase-field contour for different stages of the analysis for each of the discretization strategies is shown in Fig. 8. It is important to note that the final trajectory obtained in the different analyses is similar, characterized by the vertical propagation of the initial crack until it reaches the interface between the materials, then following along it. However, the propagation occurred in distinct steps of the analysis for each of the models, being notably slower in the edge-based T6/3 strategy. This can be justified by the fact that the T6/3 scheme generates larger support domains at the integration points, bringing a more diffuse character to the meshfree approximation, as noted by Novelli et al. [3]. After the last step presented, all models



(a) (b) (c) (d)

Figure 5. Phase-field contour plot: (a) Step 20; (b) Step 22; (c) Step 25; (d) Step 30.

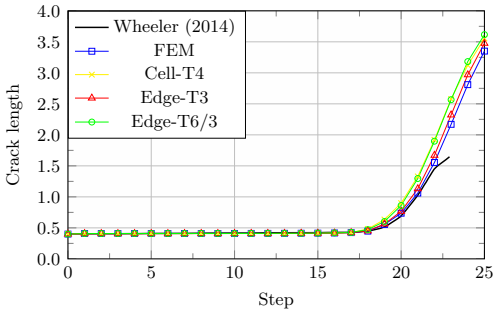
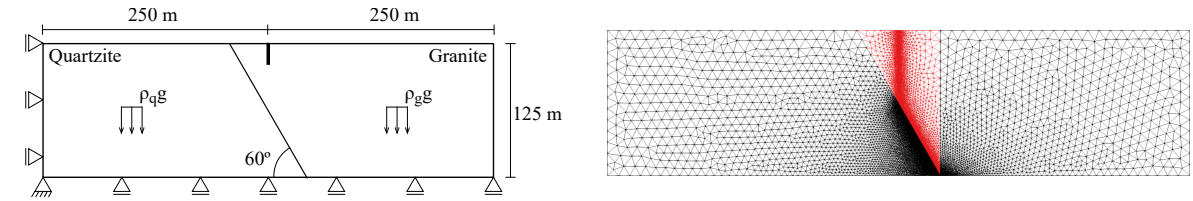


Figure 6. Development of the length of the pressurized crack throughout the analysis for the different strategies studied and comparison with the results presented by Wheeler et al. [6].

showed instabilities due to the combination of high pressure load and self-weight.

Table 1 shows a summary of the total number of global iterations performed for each of the discretization strategies analyzed. It should be noted that global iterations are defined here as iterations that ensure the simultaneous convergence of the displacement field and phase field. As the analyses evolved through different steps, an average of the number of iterations per analysis step is also presented, showing that, for this example, the edge-based T6/3 strategy proved to be more efficient in terms of convergence. The other coupled strategies presented a



(a) (b)

Figure 7. Heterogeneous quartzito-granite rock: (a) Problem setting; (b) Mesh with domain division in SPIM (in red) and FEM (in black).

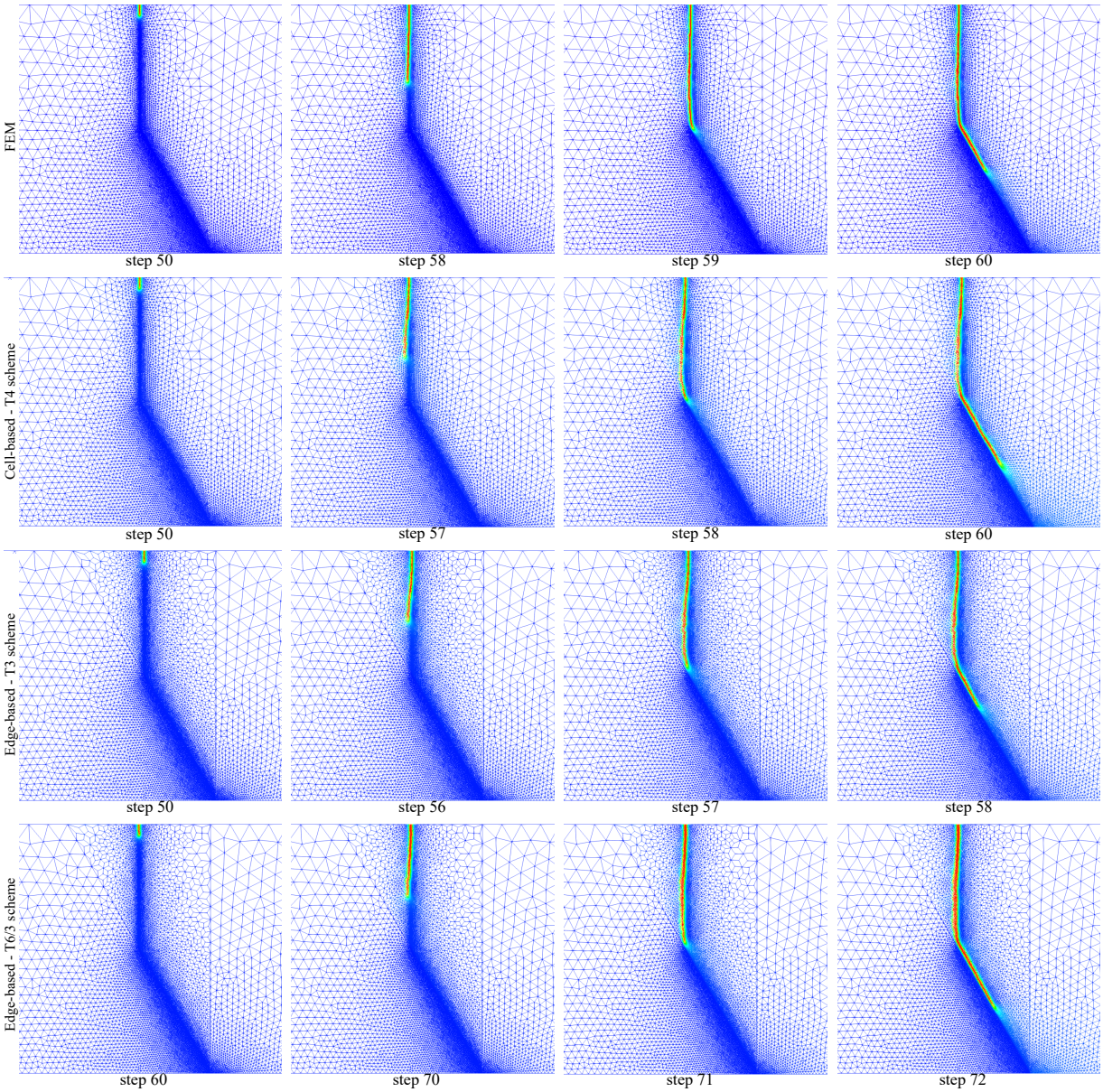


Figure 8. Phase-field contour plot for the bi-material test.

similar number of iterations per step, close to that recorded by the FEM.

Strategy	Total number of global iterations	Average number of global iterations per step
FEM	325 (60 steps)	5.42
Cell-based - T4	336 (60 steps)	5.60
Edge-based - T3	325 (58 steps)	5.60
Edge-based - T6/3	327 (72 steps)	4.54

Table 1. Global iterations necessary for convergence in the different discretization strategies.

4 Conclusions

This paper presented comparisons between smoothing domains and support node selection schemes used in the SPIM discretization of the phase field model for pressurized fractures. The model studied includes FEM-SPIM coupling, so that only the region where crack propagation is expected is discretized by SPIM. This coupling is performed at the beginning of the analysis, which requires an estimate of the crack path. The results presented validate the model, showing good agreement with those presented in the literature. The framework studied was able to successfully simulate problems with internal pressure loads, combined or not with dead loads. The first example (3.1) showed the model's ability to represent the bifurcation phenomenon, and the different strategies produced similar results. The growth of the pressurized crack occurred rapidly after reaching the critical energy G_c, as noted by Wheeler et al. [6]. The second example (3.2) showed small differences in relation to the start of propagation between the strategies, with the edge-based with T6/3 scheme lagging behind the others. However, the crack trajectory was the same in all analyses, running along the interface between the materials, through the material with lower stiffness. A comparison of the average number of iterations per step for each strategy revealed greater efficiency of the edge-based T6/3 scheme, whereas the remaining strategies showed comparable averages. Thus, it can be concluded that the FEM-SPIM coupling produced good results in the application for pressurized cracks using the phase-field model. In general, the results obtained using the different strategies studied were similar, with those from edge-based with T3 being closer to the ones from FEM discretization. A difference in the onset of crack propagation for the T6/3 scheme was found in one of the examples, requiring further investigation to determine whether it is due to greater support domains and its influence on the overall behavior of the model.

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