XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGENEERING

# Numerical modeling of the dynamic response of a beam under moving load

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**Abstract.** Bridges are fundamental elements of any nation's infrastructure. The dynamic behavior of the structure directly influences user comfort and the design limit states. This article analyzes the dynamic response of a typical bridge beam, considering traffic in one direction. The simulation follows the passage of a moving load assuming it travels through the center of the structure and, therefore, distributes the forces evenly across the supports. At the moments of interest, during the load's transit, successive integrations of the bending moment and rotation angle equations are performed, leading to the elastic curve of the beam under the assumption of linear geometric hypothesis. The approximate elastic line equation is assumed as the vibration shape. On the deformed configuration, the Rayleigh method is applied to determine the beam's natural frequency. Then, the temporal displacements generated by the passage of the load are obtained, considering a typical damping ratio for concrete structures.

**Keywords:** bridges, numerical solution, moving load, approximate elastic line, natural frequency of vibration.

## 1 Introduction

Bridges are essential structural elements in the context of a country’s road infrastructure. These structures require analysis that consider not only the static effects of loading but also the dynamic behavior in relation to the passage of vehicles. Henchi et al. [1] highlights the importance of understanding the response of bridges under moving loads at different speeds, considering the development of high-speed transportation systems and the use of lightweight materials, which have enabled the construction of bridges with large spans and reduced weight. This has often led structural dynamics research into the high-frequency spectrum.

Michaltsos, Sophianopoulos, and Kounadis [2] studied the dynamic response of simply supported beams subjected to the action of a moving load with non-negligible mass traveling at constant speed. The authors concluded that the mass and speed of the moving load were determining factors for structural displacements. Chiarelli [3] mentions the need to create analysis methods capable of inferring the durability of structures subjected to moving loads. It was observed that the response to this type of loading is associated with detrimental stresses that occur during structural use. Xu and Li [4] emphasize the importance of boundary conditions in the dynamic response of bridges. Since the supports are in contact with the ground, they become a means of vibration transmission and dissipation. In this context, Songsuwan, Pimsarn, and Wattanasakulpong [5] investigated beams made of different materials subjected to moving loads and supported on elastic foundations. It was verified that the spring constant used to represent the foundation, the length-to-height ratio of the beam, the boundary

conditions, the excitation frequency, and the speed of the load influenced the structure's natural frequencies and deformed shape. In turn, Wang, Zhu, and Lou [6] developed a nonlinear analysis model for a beam under moving load using the differential equations of motion. Ebrahimi-Mamaghani, Sarparast, and Rezaei [7] studied analytically and numerically the dynamic behavior of a functionally graded material beam aiming to evaluate vibration control parameters. This was done by applying a moving load traveling at uniform speed.

Within the context previously mentioned, this study analyzed the dynamic response of a typical bridge beam under the action of an accelerating moving load, traveling along the structure in a single direction. The simulation follows the load as it moves across the entire structure, assuming it travels along the centerline of the beam, thereby distributing the forces evenly to the supports. To determine the dynamic response, the approximated elastic line equation was assumed as the vibration mode shape. At specific moments during the force transit, successive integrations of the bending moment and rotation angle equations were carried out, leading to the elastic deformation of the beam under the assumption of linear geometry.

It is important to note that in Brazil, the structural arrangement typically adopted in bridge design consists of longitudinal girders supporting the deck and transverse beams providing lateral bracing to the main girders. The analysis of the longitudinal girders is generally conducted using Bernoulli–Euler beam theory, since these members have long spans compared to the height of their cross section, resulting in bending as the predominant behavior under service conditions. For determining the internal forces induced by vertical loading, the passage of a time-varying force is modeled as a moving load on the system. In addition to static analysis, the dynamic response to this load must also be verified, as it directly influences user comfort and the serviceability conditions of the structure. Due to the characteristics of the beam-to-column support provided using neoprene pads, the girder can be analyzed as an isolated element. In this work, the Rayleigh method is applied after defining the deformed shape of the beam, used as the vibration mode shape to determine its natural frequency. Thus, a numerical simulation was developed to determine the beam configuration under loading at each increment of the load position. To this end, the bending moments and rotation angles were calculated, which are necessary to obtain the deformed shape of the beam and its natural vibration frequency. The time-dependent displacements generated by the passage of the load were obtained assuming a damping ratio typical of concrete structures.

It is worth highlighting that the key contribution of this study lies in the fact that the modal shape of the structure is obtained from the elastic deformation of the beam for each loading configuration. As a result, the natural frequency becomes a variable that directly depends on the position of the moving load. This behavior contrasts with traditional approaches, as it considers the accumulated effect of the loads at both the current and subsequent positions at each moment of motion. By adopting a dynamic and progressive approach, the model more accurately reflects real conditions observed in highway bridges, where the position and distribution of forces influence the vibratory motion of the structure.

# 2 Description and mathematical representation of the problem

## 2.1 Static analysis

The static analysis of the problem refers to the calculation of the elastic line of the beam, that is, its deformed shape. According to Wahrhaftig et al. [8], the elastic line is the geometric position of the points of the axis when bent. Consider Fig. 1(a), which represents a beam in equilibrium under the action of a force $P$ transverse to its axis, moving from $A$ to $B$. The beam has a flexural rigidity of $EI$ and length $L$, as indicated. The origin of the reference system is located at support $A$. The movement of $P$ is defined by the variation of its position in the $x$ direction ($0 \leq x \leq L$) as a function of time, being designated as $x_p(t)$, as defined by eq. (1),.

$$x_p(t) = x_0 + v_0 t + \frac{at^2}{2} \tag{1}$$

To determine the elastic line of the beam, it is necessary to formulate the problem in relation to a position before and after the load position, represented by two arbitrary sections $\alpha$ and $\beta$, both located at a distance $x$ from the origin of the reference system and valid within their respective domains. The essential parameters for solving the elastic curve of the beam arise from the rotation and displacement conditions at the supports and at the boundaries of sections $\alpha$ and $\beta$, as illustrated in Fig. 1(b).

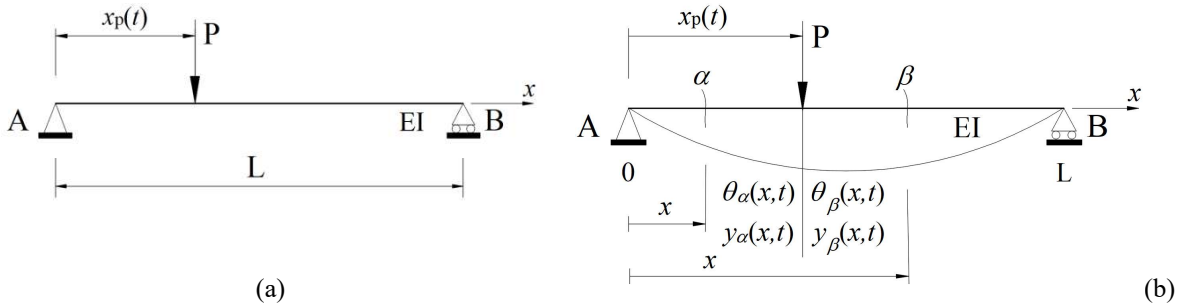



Figure 1. Representation of the problem (a) and the parameters for the beam's elastic line under moving load (b)

where $x_0$ and $v_0$ are the initial position and velocity, and $a$ is the acceleration of the force $P$. In the configuration shown in Fig. 1(b), after calculating the support reactions for the motion of $P$, the bending moment for sections $\alpha$ e $\beta$ is obtained by eqs. (2) and (3), respectively:

$$M_\alpha(x,t) = \frac{P\left(L - x_p(t)\right)x}{L} \qquad \{x \in \mathbb{R} : 0 \le x < x_p(t)\} \tag{2}$$

$$M_\alpha(x,t) = \frac{P\left(L - x_p(t)\right)x}{L} - P\left(x - x_p(t)\right) \qquad \{x \in \mathbb{R} : x_p(t) \le x \le L\} \tag{3}$$

Integrating the bending moment equation for sections $\alpha$ and $\beta$ twice with respect to $x$, the rotation and vertical displacement equations are obtained within the domains of the respective intervals, with the integration constants defined by the boundary conditions of the problem, given by:

$$x = 0 \rightarrow y_\alpha = 0\,;\, x = L \rightarrow y_\beta = 0\,;\, x = x_p(t) \rightarrow y_\alpha(x,t) = y_\beta(x,t)\,,\, \theta_\alpha(x,t) = \theta_\beta(x,t) \tag{4}$$

With the problem formulated in the preceding terms, the bending moment, the rotation of the sections and the elastic of the beam are determined by the conditions imposed in eq. (5).

$$M(x,t),\theta(x,t),y(x,t) = \begin{vmatrix} M_\alpha(x,t),\theta_\alpha(x,t),y_\alpha(x,t) & \{x \in \mathbb{R} : 0 \le x < x_p(t)\} \\ M_\beta(x,t),\theta_\beta(x,t),y_\beta(x,t) & \{x \in \mathbb{R} : x_p(t) \le x \le L\} \end{vmatrix} \tag{5}$$

In the analyses, five instants of interest were defined, designated by $t_0$, $t_1$, $t_2$, $t_3$ and $t_4$, which correspond to the positions 0, $L/4$, $L/2$, $3L/4$ and $L$ along the beam, Fig. 2(a). It is assumed that the movement starts at the origin, with the object starting from rest and accelerating to 2 m/s$^2$. Therefore, $t_0 = 0$, $t_1 = 3.50$ s, $t_2 = 4.95$ s, $t_3 = 6.06$ s and $t_4 = 7.00$ s. The trajectory of the object can be seen in Fig. 2(b).

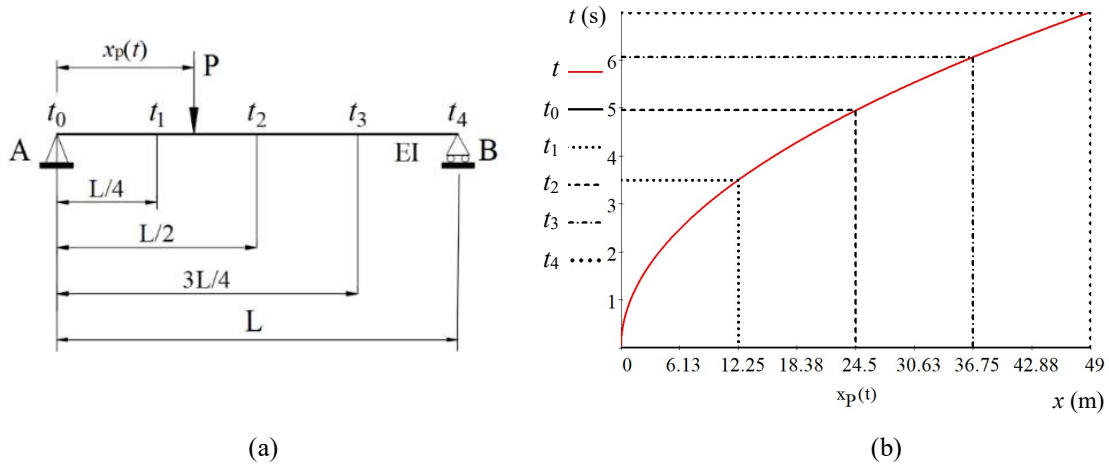
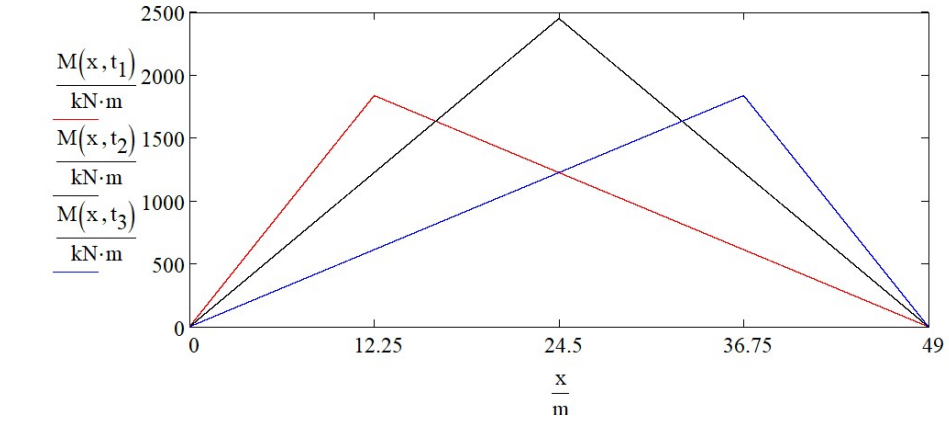
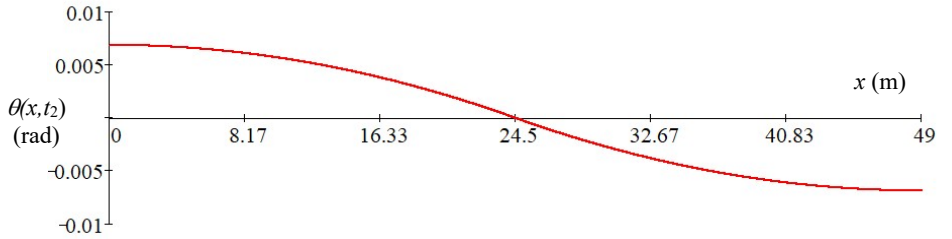



Figure 2. Positions of interest defined on the beam (a) and trajectory of the mobile load on the beam (b)

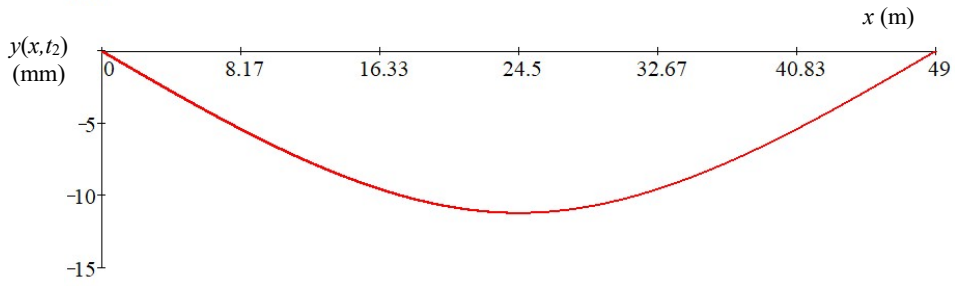
A beam 49 meters long, with a modulus of elasticity of 28160.54 MPa, a cross-sectional inertia of 15468750 cm$^4$, subjected to a force $P$ corresponding to 200.17 kN was considered. For these values, the bending moment acting on the beam can be seen in Fig. 3(a), in Cartesian coordinates. The rotation and displacement at $t_2$ are presented, as an example, in Figs. 3(b) and 3(c), respectively. It is important to note that at instants $t_0$ and $t_4$ the movable member is on the supports, producing no effects on the deformation of the beam.




(a)




(b)




(c)

Figure 3. Diagram of the bending moment for the whole beam (a), rotation (b) and displacement (c) for $t = t_2$

## 2.2 Analysis of dynamics

The analysis of the problem's dynamics begins by defining the structural stiffness and mass of the system. For this purpose, the Rayleigh method [9] is employed. In the context of the present work, both properties depend on the beam's elastic equation assumed as a shape function in the mathematical model. This implies in assuming a linear hypothesis for displacements in analysis. Rayleigh method is based on the principle of conservation of energy, and it has been used to determine the generalized stiffness and mass that allow calculating the natural frequencies of vibration of mechanical systems. The accuracy of results depends on the assumed shape of vibration which must satisfy the essential boundary conditions of the problem, displacements and rotations (Ilanko *et al.* [10]).

By considering the premises of the method as seen in Wahrhaftig *et al.* [11], the flexural stiffness is given by

$$k(t) = \int_0^L EI \left[ \frac{d^2}{dx^2} \left( \frac{y(x,t)}{y_{max}} \right) \right]^2 dx \tag{6}$$

and the mass of the system by eq. (9), initially disregarding the contribution of the beam because this is a preliminary stage of design,

$$m(t) = \frac{P}{g}\left(\frac{y\left(x_p(t),t\right)}{y_{max}}\right)^2 \tag{7}$$

where $y_{max}$ is found by searching for the maximum value and is used to normalize the function. The natural frequency and damped natural frequency can be calculated, respectively, by

$$\omega(t) = \sqrt{\frac{k(t)}{m(t)}} \quad \text{and} \quad \omega_D(t) = \sqrt{1-\xi^2}\,\omega(t) \quad \text{(in rad/s)} \tag{8}$$

With the transformations proposed by A. Akay and A. Carcaterra [11], the vertical displacement for any instant of the moving load movement, for a damping ratio $\xi$, is calculated by eq. (11).

$$v(t) = y\left((x,t),t\right)e^{-\xi\omega(t)t}\cos\left(\omega_D(t)t\right) \tag{9}$$

The final vibratory motion $v_F$, defined by the superposition of the effects for each instant $t_n$ from the entry of the load into the structure until its exit, can be obtained by.

$$v_{Ftn}(t) = \sum_{i=0}^{n} v_{t(n-i)}(t) \qquad \{n \in \mathbb{N} : 0 \le n \le 4\} \tag{10}$$

## 3 Simulation, results and discussion

The system response at instants $t_1$ to $t_3$ is presented in the graphs of Figs. 4 to 6, where (a) indicates the load position, (b) the movement when the mobile leaves the position, and (c) the final displacement of the position. The movement described corresponds to the moment immediately after the mobile leaves the indicated position, eq. (11). Upon leaving it, the system vibrates in a damped motion and is influenced by the displacements of the subsequent positions. The final effect is computed by the sum of all the effects produced as the load follows its trajectory, eq. (12). The damping ratio used in analysis corresponds to 2%. A sensitive evaluation for damping ratios is shown in Table 1.

The complete response of the beam can be seen in Fig. 7. The calculation considers the frequency of the structure determined at time increments of 0.035 s between $t_0$ and $t_4$. This means that for each new instant the bending moment, the rotation, the elastic force of the beam, the structural stiffness, the mass of the system, the undamped and damped natural frequency, and the structural displacement are calculated according to eq. (11).

### 3.1 Moving load at $t = t_1$

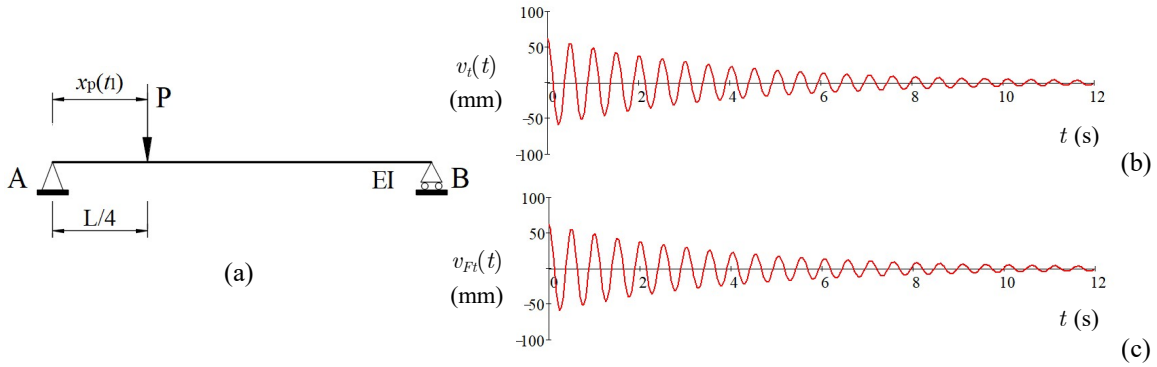



Figure 4. Vibratory movement for $t = t_1$.

### 3.2 Moving load at $t = t_2$

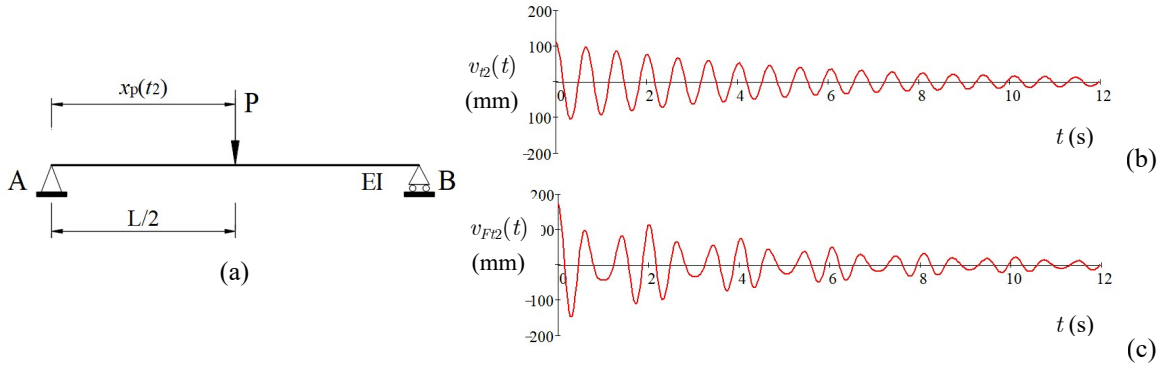



Figure 5. Vibratory movement for $t = t_2$.

### 3.3 Moving load at $t = t_3$

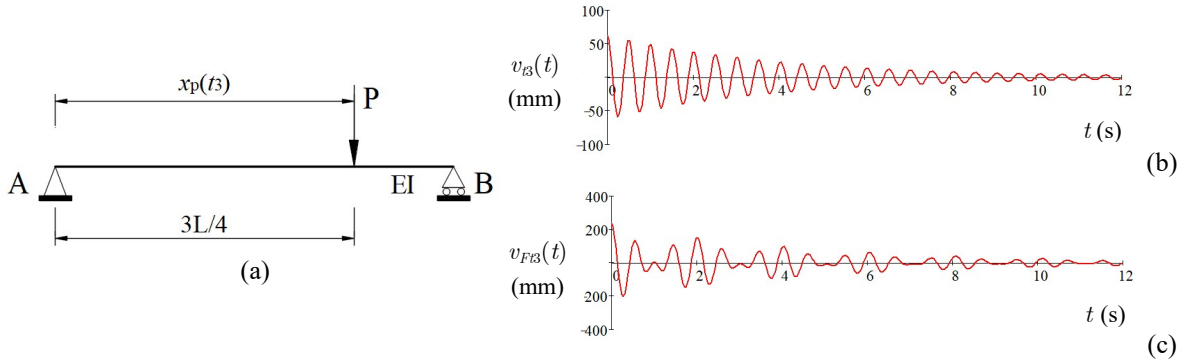



Figure 6. Vibratory movement for $t = t_3$.

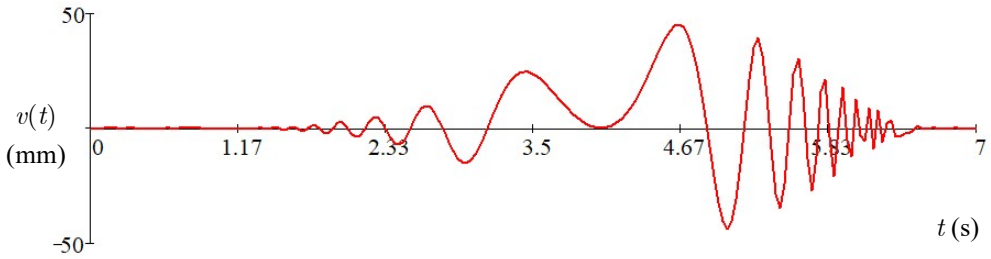



Figure 7. Response of the beam in damped vibration under the effect of loading

Table 1. Sensitivity evaluation for frequency ω and displacement $v$ (ξ = damping ratio)

| Frequency ω (rad/s) | | | | | | | |
|---|---|---|---|---|---|---|---|
| ξ | $t_1$ | $t_2$ | $t_3$ | ξ | $t_1$ | $t_2$ | $t_3$ |
| 0% | 12.442 | 9.331 | 12.442 | 10% | 12.379 | 9.284 | 12.379 |
| 2% | 12.439 | 9.329 | 12.439 | 15% | 12.301 | 9.226 | 12.301 |
| 5% | 12.426 | 9.319 | 12.426 | 20% | 12.190 | 9.143 | 12.190 |
| Displacement $v$ (mm) | | | | | | | |
| $v$ | 63.354 | 112.629 | 63.354 | $v_{Fmax}$ | 63.354 | 175.983 | 239.336 |

# 4 Conclusions

This work performed a numerical simulation to analyze the dynamic behavior of a typical bridge girder subjected to a moving load, using a damping ratio commonly used in concrete structure design. The Rayleigh method was applied to determine the beam's natural frequency. The numerical simulation began by defining the beam's configuration under the effect of the load at each increment in the load position, with subsequent calculation of the structural response. Four different beam positions were analyzed, established for the instants of interest during the force passage, in addition to the overall response of the structure. Based on the results obtained, the following can be indicated:

- The numerical method used proves suitable for predicting the dynamic behavior of structures under variable load, considering the structural damping ratio, the initial velocity, and the acceleration of the moving load on the structure.
- The final vibratory motion should be obtained by superimposing the effects produced at the current positions ahead of the one the mobile has just left. The displacements in each point along the beam axis depend on the load position during its trajectory, and the flexural rigidity *EI.* The maximum ratio between the superposed displacement in a position (in this case $t_3$) and the initial one is 4 times;
- Displacement compliance analysis is subject to the requirements of the technical standards that guide specific design practices;
- The method of analysis and the results obtained can support actions to formulate measurements of control of vibrations.

Future work with simulations predicting the performance of bridge structures under different load configurations and traffic conditions are awaited. Comparisons with other mathematical processes are also desired.

**Acknowledgements**. The authors would like to thank PIBIC/UFBA, the Institutional Program for Scientific Initiation Scholarships of the Federal University of Bahia, and CNPq (National Council for Scientific and Technological Development) for the scholarships provided.

**Authorship statement.** The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

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