

A Non Uniform Algebraic Dynamic Multilevel for Unstructured Meshes for the Numerical Simulation of Fluid Flow in Heterogeneous Porous Media

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Abstract. In the present work, we introduce, for the first time in the literature, the Non-Uniform Algebraic Dynamic Multilevel with Unstructured Meshes (NU-ADM-U) method, which extends the original Non-Uniform Algebraic Dynamic Multilevel method to the simulation of two-phase flows on unstructured meshes. In this new method, the employed multiscale transfer operator is the Algebraic Multiscale Solver for General Unstructured Grids (AMS-U). To construct the coarse meshes (primal and dual), we adopted the procedure proposed by AMS-U method. For the pressure field, we adopted a cell-centered finite volume formulation of the MPFA-D (Multi-Point Flux Approximation with a Diamond stencil) type, with weights computed using the Global Least Squares (GLS) method. This formulation can handle non-K-orthogonal meshes and yields highly accurate solutions and enables the use of unstructured meshes at both resolution levels. The results showed that the local refinement guided by the new parameter was able to accurately represent the fine-scale pressure field, reducing spurious oscillations and, consequently, the associated errors, while also decreasing the number of active volumes.

Keywords: Reservoir Simulation; Finite Volume Method, Multiscale-Multilevel Method, Unstructured Mesh.

1 Introduction

Oil and gas production is an activity that requires substantial investments and is associated with high financial risks. Therefore, the ability to predict the behavior of a petroleum field through numerical simulations is essential. This approach enables the approximation of production curves and the anticipation of potential risks, which is crucial for decision-making given the high financial and environmental impact inherent to this industry. In most simulation software, the Two Point Flux Approximation (TPFA) finite volume formulation is adopted for flow approximation. However, this discrete formulation converges to the correct solution only when the grid directions are aligned with the principal directions of the permeability tensor [1]. In contrast, Multi-Point Flux Approximation (MPFA) flow approximations correctly discretize the flow equations for non-orthogonal grids as well as for general orientations of the permeability tensor's principal directions. Moreover, they offer greater flexibility for modeling complex geological features and can readily accommodate mesh adaptation [2].

A typical reservoir contains approximately 10^6 to 10^9 cells (control volumes) in the computational domain, and depending on the employed flow model, the number of degrees of freedom can increase significantly. Moreover, with technological advancements, the amount of geological data acquired from reservoirs has grown substantially, enabling higher-resolution models [3]. However, high-resolution models are rarely used in reservoir modeling due to computational limitations, which make direct simulation impractical [3]. To overcome this, the upscaling method is commonly employed, whereby coarse-scale properties are obtained using analytical or numerical homogenization methods with appropriate boundary conditions [4]. Alternatively, multiscale methods employ scale transfer operators to solve the problem at the coarse scale and subsequently obtain the fine-scale solution through scale transfer simple algebraic operations. The coarse scale is used to accelerate the simulation, while

fine-scale information is preserved [5], thereby improving the solution quality compared to traditional upscaling methods.

In this context, the present work introduces the Non-Uniform Algebraic Dynamic Multilevel method for Unstructured grids (NU-ADM-U), which extends the original NU-ADM [6] formulation. The new approach employs the Multi-Point Flux Approximation with Diamond stencil (MPFA-D) at the fine scale and Algebraic Multiscale Solver for Unstructured grids (AMS-U) operators at the coarse scale. To enhance solution quality and guide the mesh adaptation process, two auxiliary error indicators are introduced. Our new method has been evaluated using two-phase benchmark tests and was able to accurately approximate the fine-scale flow patterns.

2 Mathematical model

The single-phase flow model in porous media is derived from the mass conservation law and Darcy's law for fluid velocity in porous media. Neglecting the effects of gravity, rock and fluid compressibility, as well as the assumptions under which Darcy's law holds, the governing equation is respectively given by [7]:

$$-\nabla \cdot \mathbf{v} = Q, \quad (1)$$

$$\mathbf{v} = -\mathcal{K}\nabla p, \quad (2)$$

where \mathbf{v} is the fluid velocity, Q is the source term, \mathcal{K} is the absolute rock permeability tensor and p is the pressure.

3 Numerical approximation

By applying the Finite Volume Method (FVM) in the fine-scale control volumes, where the fine-scale fluxes are calculated by the MPFA-D method, where the local stencil illustrated in Fig. 1, and the nodal interpolation is obtained by the Global Least Squares (GLS) method [8], making the flux approximation fully cell-centered, we obtain the linear system at the fine-scale:

$$\mathbf{T}^f \mathbf{p}^f = \mathbf{q}^f, \quad (3)$$

where \mathbf{p}^f are the cell pressures unknowns, \mathbf{T}^f is the transmissibility matrix and \mathbf{q}^f is the RHS term, all defined at the fine-scale.

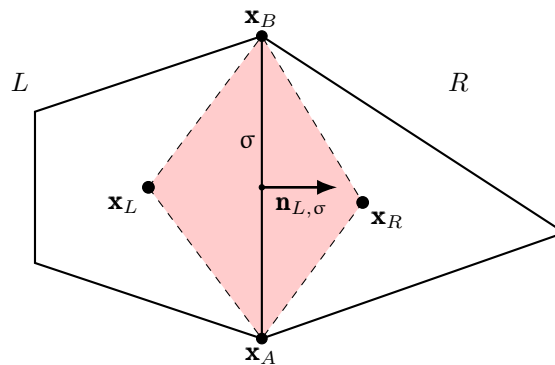


Figure 1. Diamond stencil used for edge-based flux computation, highlighted in light red. $\mathbf{n}_{L,\sigma}$ is the unitary normal vector to σ pointing outward from control volume L , \mathbf{x}_L and \mathbf{x}_R are the centroids of control volumes L and R , \mathbf{x}_A and \mathbf{x}_B are the geometric coordinates of A and B nodes and σ is the interface between L and R , where $|\sigma| = |\mathbf{x}_B - \mathbf{x}_A|$ is the length of σ .

4 Non Uniform Algebraic Dynamic Multilevel method for Unstructured Grids

4.1 AMS-U method

The Multiscale Finite Volume Method (MsFVM) is constructed using two coarse grids: the primal (Ω^p) and the dual (Ω^d) coarse meshes [9]. Based on these meshes, two scale-transfer operators are defined: the prolongation operator (P), responsible for downscaling the coarse-scale solution to the fine scale, and the restriction operator (\mathcal{R}), which performs the upscaling. More recently, Souza et al. [1] introduced the Algebraic Multiscale Solver for Unstructured Grids (AMS-U), which extends the conventional AMS framework [10] to handle unstructured discretizations. In this approach, the concept of a background grid is employed to enable a flexible construction of both primal and dual coarse grids, leveraging the connectivity information of fine-scale control volumes.

The primal coarse grid is generated based on a background coarse mesh, which delineates the spatial extent associated with each primal control volume. The construction algorithm ensures that all fine-grid cells belonging to a given primal coarse cell are interconnected via shared interfaces. An example of a resulting primal coarse grid is depicted in Fig. 2c, obtained from the background mesh shown in Fig. 2b and the fine-scale grid illustrated in Fig. 2a.

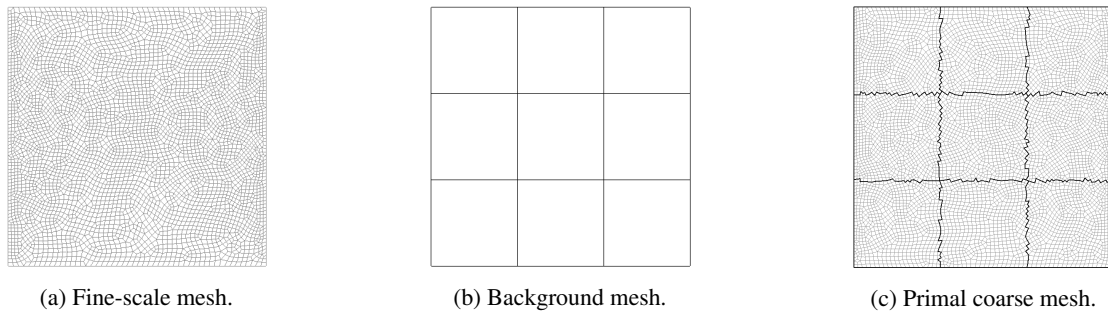


Figure 2. Unstructured primal coarse mesh generation.

The construction of the dual coarse grid (Fig. 3a) follows the definition of the primal coarse grid. The path of each dual edge, linking the dual vertex (i.e. the fine scale element closer to the centroid of the primal coarse mesh) to the centroid of a coarse interface, is computed using Dijkstra's algorithm [1]. This approach guarantees that the connection path traverses physically connected fine-scale elements via their shared interfaces. Figure 3c illustrates the construction of the support region based on the interaction region of the dual mesh vertex (Fig. 3b).

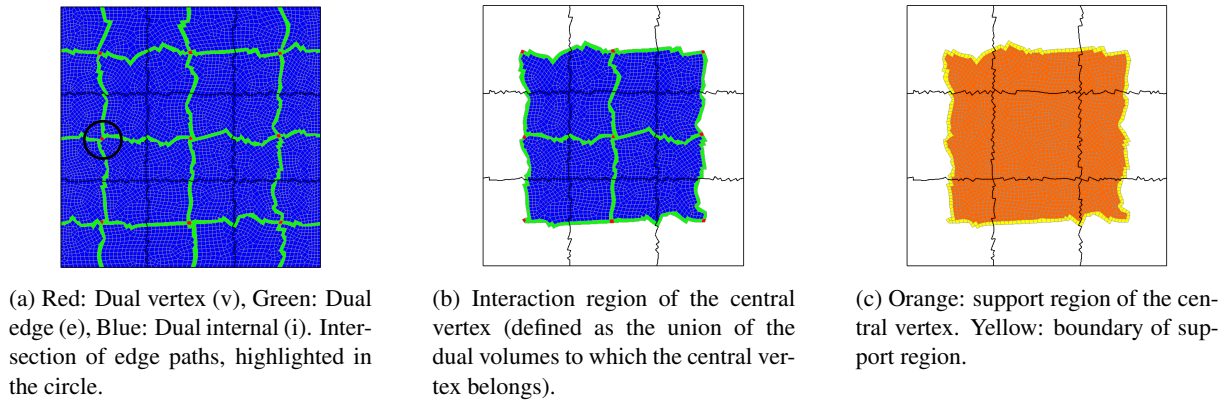


Figure 3. Unstructured dual coarse mesh and construction of the support region.

The prolongation operator is constructed by computing basis functions φ_i^I that are obtained by solving localized problems using Reduced Boundary Conditions (RBC) [11]. Accordingly, the fine-scale pressure is approximated as a linear combination of the coarse-scale pressure values using the prolongation operator, which is defined as follows [6]:

$$\mathbf{p}^f \approx \mathbf{p}^{ms} = \sum_{I=1}^{N(\Omega^p)} \varphi_i^I p_I^c = P \mathbf{p}^c, \quad (4)$$

where \mathbf{p}^{ms} is the multiscale pressure, $N(\Omega^p)$ is the number of coarse cells and \mathbf{p}^c is the coarse pressure vector.

Following the AMS methodology, the prolongation operator is defined by:

$$\mathbf{P} = \mathbf{G}^T \begin{bmatrix} \mathbf{T}_{ii}^{-1} \left(\mathbf{T}_{ie} \tilde{\mathbf{M}}_{ee}^{-1} \mathbf{T}_{ev} - \mathbf{T}_{iv} \right) \\ - \tilde{\mathbf{M}}_{ee}^{-1} \mathbf{T}_{ev} \\ \mathbf{I}_{vv} \end{bmatrix} = \mathbf{G}^T \begin{bmatrix} \mathbf{P}_{iv} \\ \mathbf{P}_{ev} \\ \mathbf{P}_{vv} \end{bmatrix} \quad (5)$$

where \mathbf{G} is the permutation matrix that maps the fine-scale ordering to wirebasket ordering, \mathbf{T}_{ab} , with $a, b \in \{v, e, i\}$, refers to the local influence of b volumes on a volumes and $\tilde{\mathbf{M}}_{ee}$ is the modified local matrix obtained by applying the reduced boundary conditions and eliminating the contributions from internal-classified volumes to edge-classified volumes [10]. The AMS-U prolongation operator is obtained by imposing zero values on the basis functions located at the boundary of the support region of the dual mesh vertex, where basis function leakage may occur due to the intersection of two dual-edge paths [1]. To obtain the AMS-U prolongation operator, once the basis functions have been computed along the edges of the dual mesh, it is necessary to restrict them to the corresponding support region only, as already mentioned, i.e.,

$$\mathbf{P}_{ev} = 0 \quad \forall e \notin \text{support region of the vertex } v, \quad (6)$$

and to ensure partition of unity:

$$\mathbf{P}_{ev}^{corr} = \frac{\mathbf{P}_{ev}}{\sum_v \mathbf{P}_{ev}}. \quad (7)$$

In this way, the complete AMS-U prolongation operator, and the restriction operator are obtained:

$$\mathbf{P} = \mathbf{G}^T \begin{bmatrix} -\mathbf{T}_{ii}^{-1} \left(\mathbf{T}_{ie} \mathbf{P}_{ev}^{corr} + \mathbf{T}_{iv} \right) \\ \mathbf{P}_{ev}^{corr} \\ \mathbf{I}_{vv} \end{bmatrix}, \quad \mathcal{R}_{Ii} = \begin{cases} 1 & \forall \Omega_i^f \in \Omega_i^p \\ 0 & \text{otherwise} \end{cases}. \quad (8)$$

4.2 Refinement Criteria

To accurately capture regions exhibiting high permeability contrasts and to identify zones with high pressure gradients, we employ the parameter γ defined as follows:

$$\gamma = \frac{|\nabla_{p_L} - \nabla_{p_R}|}{|\sigma|}, \quad (9)$$

where ∇_{p_L} and ∇_{p_R} are the gradients evaluated in the volumes adjacent to the interface σ and $|\sigma|$ is the length of σ . If $\gamma > \gamma^{lim}$, where γ^{lim} is a user-defined threshold, the sets $\mathcal{C}_A \cup \mathcal{C}_B$ are maintained at the fine-scale, where A and B are the vertices sharing the interface σ , and \mathcal{C}_A and \mathcal{C}_B are the fine-scale control volumes associated with vertices A and B , respectively.

Furthermore, dual coarse grids volumes that contain prescribed values in the fine grid are retained at fine scale in order to enhance solution accuracy in these regions.

4.3 NU-ADM-U solution

Once the multiscale operators are obtained, the NU-ADM-U operators are defined as follows:

$$\mathbf{P}_{ij}^{NU-ADM-U} = \begin{cases} \mathbf{P}_{iI} & \text{if } \Omega_i^f \in \Omega^p \\ \delta_{ij} & \text{otherwise} \end{cases}, \quad \mathcal{R}_{Ji}^{NU-ADM-U} = \begin{cases} \mathcal{R}_{Ii} & \text{if } \Omega_i^f \in \Omega^p \\ \delta_{Ji} & \text{otherwise} \end{cases}, \quad (10)$$

where J is the ID on NU-ADM mesh, δ is the Kronecker delta function and Ω^p is the coarse-scale domain. The NU-ADM-U pressure solution ($\mathbf{p}^{NU-ADM-U}$) is given by:

$$\mathbf{p}^{NU-ADM-U} = \mathbf{P}^{NU-ADM-U} (\mathcal{R}^{NU-ADM-U} \mathbf{T}^f \mathbf{P}^{NU-ADM-U})^{-1} \mathcal{R}^{NU-ADM-U} \mathbf{q}^f. \quad (11)$$

5 Results

The NU-ADM-U pressure field is used as approximated solution, the fine scale solution is considered the reference solution and the error norms are defined as:

$$E_{L_2}(\mathbf{X}^{ref} - \mathbf{X}^{approx}) = \left(\sum_i \left(X_i^{ref} - X_i^{approx} \right)^2 \div \sum_i \left(X_i^{ref} \right)^2 \right)^{1/2}, \quad (12)$$

$$E_{L_\infty}(\mathbf{X}^{ref} - \mathbf{X}^{approx}) = \frac{\|\mathbf{X}^{ref} - \mathbf{X}^{approx}\|_\infty}{\|\mathbf{X}^{ref}\|_\infty},$$

where \mathbf{X}^{ref} and \mathbf{X}^{approx} are the reference and approximated solution. This example, adapted from [1], considers single-phase flow in a quarter five-spot configuration within a unit square domain, $\Omega = [0, 1] \times [0, 1]$. This problem is commonly modeled as a two-phase flow rather than a single-phase flow because water–oil displacement more realistically represents secondary recovery, where fluid interactions, mobility ratio, and relative permeability are key to capturing the flow behavior. The permeability field consists of an outer zone with higher permeability and an inner barrier, shaped as a Maltese cross, with lower permeability. No-flow boundary conditions are imposed on the domain, with a prescribed injection flow rate ($Q_I = 2$) at the injection well located in the top-left corner and a prescribed pressure ($p_P = 1$) at the production well located in the bottom-right corner.

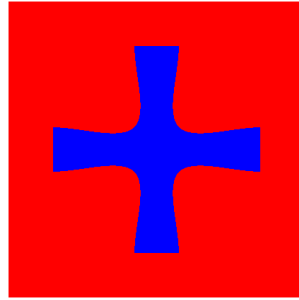


Figure 4. Permeability field. Red = \mathcal{K}_1 and blue = \mathcal{K}_2 .

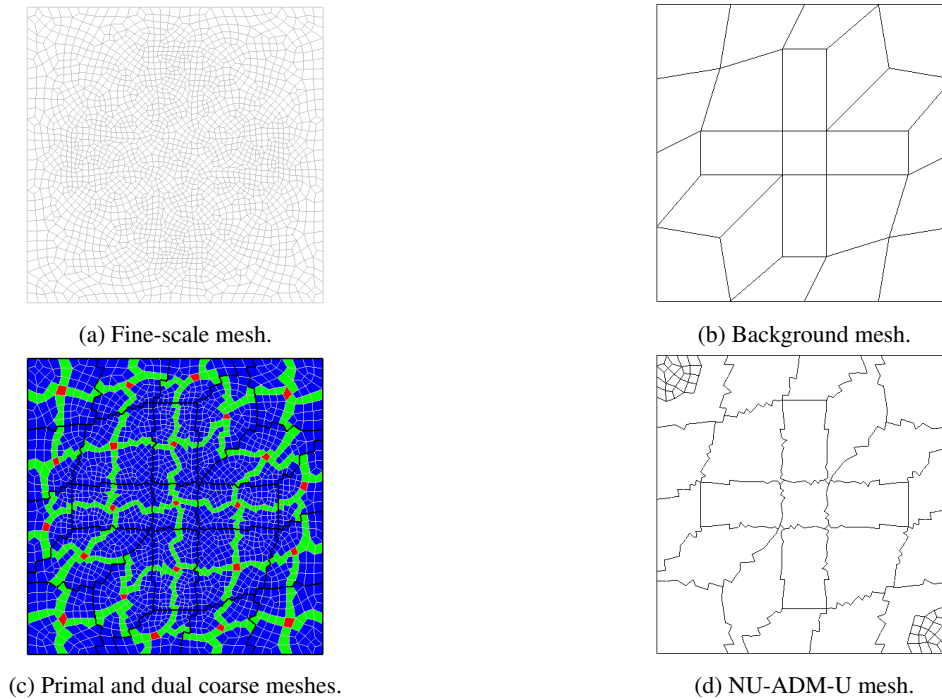


Figure 5. Meshes employed.

The permeability field is depicted in Fig. 4, where the Maltese cross pattern appears in the interior region. The fine unstructured mesh, consisting of 2,276 quadrilateral elements, along with the coarse primal, dual, and background meshes adapted to the permeability field, are shown in Figs. 5a to 5c. In this example, the Maltese cross region is assigned a low permeability value (\mathcal{K}_2), while the exterior region has a permeability \mathcal{K}_1 , as defined as in eq. (13).

$$\mathcal{K}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{K}_2 = \begin{bmatrix} 1E-3 & 0 \\ 0 & 1E-3 \end{bmatrix}. \quad (13)$$

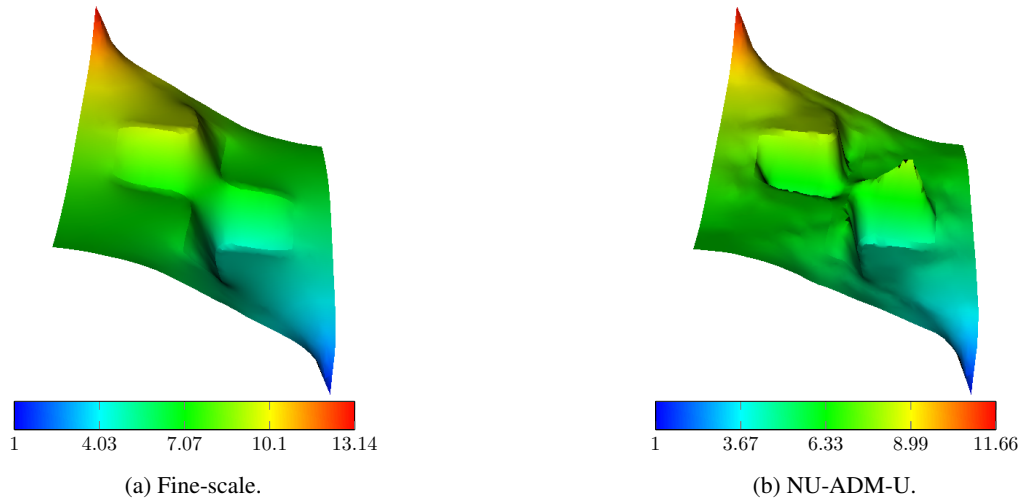


Figure 6. Pressure fields.

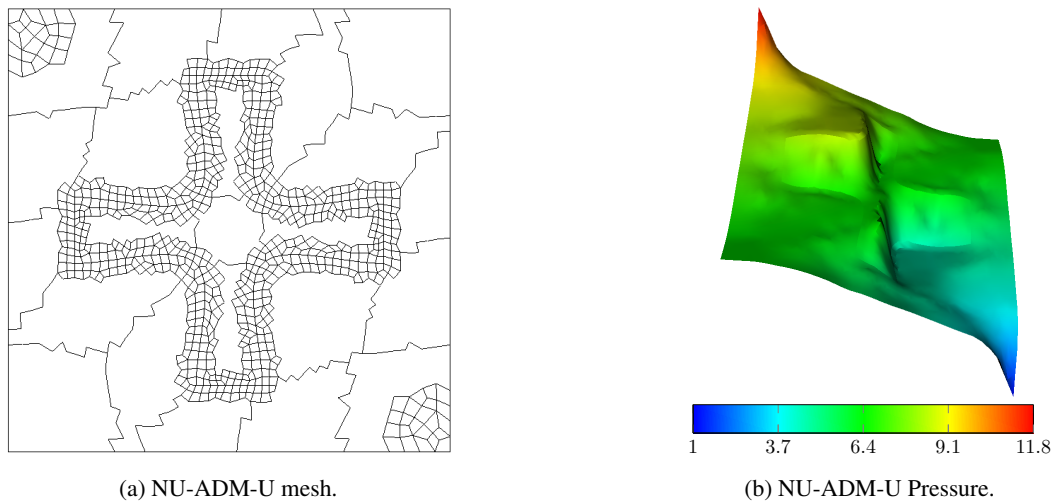


Figure 7. NU-ADM-U mesh (a) and pressure field (b) using $\gamma^{lim} = 10$.

Initially, only the coarse dual mesh volumes with prescribed flow or pressure conditions (i.e., those containing the wells) are retained at the fine scale (see Fig. 5d). With this initial NU-ADM-U mesh configuration, the pressure field exhibits oscillations in the interface regions between different permeability zones, as observed in the comparison between the fine-scale pressure field (Fig. 6a) and the NU-ADM-U mesh solution (Fig. 6b). The NU-ADM-U mesh generated using the γ estimator, with $\gamma^{lim} = 10$ along with the corresponding pressure field, are shown in Figs. 7a and 7b.

From Table 1, it can be observed that, in addition to improving solution quality by reducing $E_{L_2}(p)$ by 23%, the value of $E_{L_\infty}(p)$ decreased by 58%, accompanied by an approximate 30% increase in the number of active volumes. Despite small localized errors in the cross region, the NU-ADM-U solution closely approximated the reference solution, mitigating spurious oscillations in the pressure field by preserving fine-scale resolution in selected regions identified through the γ parameter.

Table 1. $E_{L_\infty}(p)$, $E_{L_2}(p)$ and % NU-ADM-U volumes.

	Initial NU-ADM-U mesh	NU-ADM-U mesh using γ
$E_{L_\infty}(p)$	0.29	0.12
$E_{L_2}(p)$	0.13	0.10
% NU-ADM-U volumes	2.94	32.64

6 Conclusions

In this paper, we introduce the Non-Uniform Algebraic Dynamic Multilevel method for Unstructured grids (NU-ADM-U), which extends the original NU-ADM formulation. The proposed method employs the Multipoint Flux Approximation with Diamond stencil (MPFA-D) scheme at the fine scale and AMS-U operators at the coarse scale. This extension enables the use of unstructured meshes at both resolution levels, thereby enhancing the method's capability to handle complex geological features. The results demonstrate that applying local mesh refinement, guided by the new parameter γ , effectively reduces non-physical pressure oscillations and the errors $E_{L_\infty}(p)$ and $E_{L_2}(p)$.

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