XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING

Simplified global geometric and physical nonlinear analysis of reinforced concrete columns considering haunch variation and regular cross-sections

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Abstract. The main objective of this work aims to present computational results related to the global nonlinear geometric and simplified physical analysis of reinforced concrete columns with haunch variation and rectangular cross-sections, considering the buckling in relation to the axis of greater inertia, through the use of an algorithm capable of using Simpson's Rule to obtain the equivalent stiffness of the element and to perform the iterative process P-Δ for the determination of slender column design demands. It is worth mentioning that these demands serve as a key element for the realization of a safe and cost-effective design of the structures. In order to validate the results obtained, the values of the 2nd order bending moments were compared with the answers determined from the column stability criterion (Gamma z coefficient) recommended by ABNT NBR 6118:2023 [1]. The methodology used provided consistent and more conservative results than the simplified method recommended by the Brazilian standard.

Keywords: reinforced concrete columns, nonlinear analysis, gamma Z coefficient, haunchs, numerical methods.

1 Introduction

Although other lighter construction systems are on the rise, reinforced concrete is still widely used in the construction sector, even in taller buildings. In these cases, it is essential to have an analysis that describes the behavior of the structure in a way that is closer to reality, in order to propose a safe and economical structure. Nonlinear, or second-order, analysis is encouraged by current design standards, as it considers the balance of the structure in a deformed configuration, allowing the behavior of the structure to be verified in a more realistic way. Despite this, it is still common for design engineers to use linear-elastic analysis, because nonlinearity, or second-order effects, lead to more laborious, time-consuming design and analysis processes that require greater computational effort. The great disadvantage of working within the elastic-linear regime is that the material is restricted to low stresses, sometimes making the project uneconomical. In addition, knowing the real behavior of the material, especially close to the fault, allows the development of safer structures. There are different types of structural analysis of frames, namely: the 1st order analysis is one in which the material is considered elastic-linear and the balance of the structure is determined in its undisturbed condition; geometric nonlinear analysis, or 2nd order, with elastic material, in which the equilibrium of the structure is determined in the deformed condition and the material remains in the linear regime; 1st order analysis and elastoplastic behavior material, in this analysis model the undeformed structure and the material in elastoplastic regime are considered and, finally, the geometric and physical nonlinear analysis, in which the equilibrium is determined for the deformed condition of the structure and the material presents elastoplastic behavior. This last model of analysis is the one that comes closest to the real

behavior of the structure, because, according to SILVA, R. [2], the loss of rigidity due to the plasticization of the elements and the large deformations of the structure are taken into account in the design of the structure. Therefore, the objective of this research is to develop the global analysis of 2nd order of slender columns with and without haunch variation by means of an iterative algorithm and to compare the results obtained with the simplified methods of the Brazilian standard.

2 Bibliographic References

2.1 Nonlinear analysis of the structure

As highlighted earlier, the nonlinearity of structures can manifest itself in two ways: physical nonlinearity (NLF) and geometric nonlinearity (NLG). Geometric nonlinearity is related to the stability of the structure, especially of slender structures, and refers to the effects caused by its global displacement and the deformations of the constituent elements. This effect occurs due to the emergence of additional moments caused by the product between the vertical loads and the horizontal displacement of the structure – the *well-known P-Δ* effect – and also by the product between the local deformations of the structural elements and the acting axial forces – *P-δ effect*, according to SILVA, M. [3]. In these cases, according to MENDES [4], the large displacements can lead to the collapse of the structure even before the material drains.

Physical nonlinearity, on the other hand, is associated with the material that makes up the structure. Reinforced concrete is a complex material, whose cracking process causes a reduction in the rigidity of the element. Cracks occur when the tensile strength of the concrete is exceeded, with this, the area of concrete that effectively resists the forces is reduced and, consequently, the inertia of the cracked section as well, as stated by GOMES [5]. Due to this, each increase in stress, after the beginning of cracking, will cause a change in the stiffness of the structural element, translated by the product EI, where E is the longitudinal modulus of elasticity of the material and I is the moment of inertia of the cracked cross-section. The process becomes, therefore, iterative and laborious, and therefore requires the help of software capable of considering the physical nonlinearity of the material in the structural design. The creep and flow of steel reinforcements also contribute to the nonlinear behavior of reinforced concrete [1] and PINTO [6].

The ABNT NBR 6118:2023 [1], however, allows, in a simplistic way, the consideration of an approximation of the physical nonlinearity of reticulated structures, with at least four floors, to carry out the analysis of the 2nd order global forces, through approximate values of the stiffness of the structural elements, the secant stiffness, for columns $(EI)_{sec} = 0{,}8E_cI_c$, where I_c is the moment of inertia of the raw concrete section, including, where applicable, the collaborating tables and E_c is the representative value of the concrete deformation modulus. The approximations cannot, however, be used for 2nd order local purposes. In this case, the isolated elements must be analyzed according to item 15.8 of the standard.

As for geometric nonlinearity, or 2nd order forces, in which the balance of the structure is considered in its deformed condition, ABNT NBR 6118:2023 [1] recommends nonlinear analysis only for structures whose horizontal displacements cause an increase in internal loads and reactions greater than 10% compared to the same values obtained according to the elastic linear analysis, or 1st order. Thus, the standard classifies structures as follows: structures of non-sway frames, in which horizontal displacements are negligible and nonlinear analysis is dispensed with, and in structures of sway frames, those in which horizontal displacements are significant and, therefore, geometric and physical nonlinear analysis must be carried out, mandatorily.

To classify the structure as non-sway frames or sway frames, ABNT NBR 6118:2023 [1], in item 15.5, presents the α instability parameter and the coefficient γ_z, the latter, in addition to classifying the structure, allows estimating the 2nd order effects by amplifying the efforts.

2.1.1 Instability Parameter (α)

According to MONCAYO [7], this parameter is suitable for verifying the overall stability of reinforced concrete structures; however, it is not able to quantify the second-order effects that directly influence the nonlinear behavior of the system.

According to ABNT NBR 6118:2023 [1], "a symmetrical lattice structure can be considered a non-sway frame of

fixed-nodes when its instability parameter α is less than the value α_1", in which α is expressed by:

$$\alpha = H_{tot}\sqrt{\frac{N_k}{E_{cs}I_c}} \tag{1}$$

and $\alpha_1 = 0{,}2 + 0{,}1n$ if $n \leq 3$, or $\alpha_1 = 0{,}6$ if $n \geq 4$. Being n the number of floors above the foundation or a level that cannot be moved from the basement; H_{tot} the total height of the structure, measured from the top of the foundation or from a slightly displaceable level of the subsoil; N_k the sum of all vertical loads acting on the structure (from the level considered for the calculation of H_{tot}), with its characteristic value and; $E_{cs}I_c$ represents the sum of the stiffness values of all the columns in the direction considered.

2.1.2 Coefficient γ_z

The parameter γ_z was established in 1991 by FRANCO and VASCONCELOS [8], this parameter allows not only to classify the structure, but to increase the efforts of the first order, thus allowing a global analysis of the structure in a quick and simple way [7].
According to NBR 6118:2023 [1], the parameter γ_z is valid for reticulated structures of at least four floors and is calculated as follows:

$$\gamma_z = \frac{1}{1 - \frac{\Delta M_{tot,d}}{M_{1,tot,d}}} \tag{2}$$

where $M_{1,tot,d}$ is the overturning moment, that is, the sum of the moments of all the horizontal forces of the combination considered, with their calculation values, at the base of the structure and $\Delta M_{tot,d}$ is the sum of the products of all the vertical forces acting on the structure, in the combination considered, with their calculation values being multiplied by the horizontal displacements of their respective points of application, obtained from the 1st order analysis.
The Standard considers that the structure is a sway frames of moving nodes when γ_z values are greater than or equal to 1.1. For these structures, the 2nd-order global forces are obtained by amplifying the horizontal actions of the combination considered to be 1st order by the factor $0.95\gamma_z$. This process is allowed for values of $\gamma_z \leq 1{,}3$.
The same standard also highlights that physical nonlinearity can be considered using the approximate values of stiffness mentioned in item 2.1.
The 2nd order stresses of slender structures, with $\gamma_z \geq 1{,}1$, also can be obtained by means of the process $P - \Delta$, which will be described below.

2.1.3. Process $P - \Delta$

The horizontal loads applied to the structure induce displacements in its nodes, which are known as 1st order displacements. According to LIMA [9], the vertical forces (P), when acting on these horizontally displaced nodes of a Δ value, generate additional moments in the structure, causing new displacements associated with 2nd order effects. This cycle repeats until the structure assumes a position of equilibrium.
The P-Δ Method consists of the calculation of the moment portions generated by the acting axial loads and the horizontal displacements of the structure. Such an approach, however, does not allow the evaluation or classification of the structural system, serving only to determine the 2nd order efforts. Thus, for the proper classification of the structure, it is first necessary to calculate the instability parameter α or the coefficient γ_z.
MONCAYO [7] cites several methods of analysis based on the P-Δ procedure: the Method of Two Iterative Cycles, the Iterative Gravity Charge Method, the Negative Stiffness Method and the Fictitious Charge Method — the latter being adopted both by the author and in this research.
The process $P - \Delta$ using the fictitious load is an iterative process in which, initially, the 1st order analysis is performed to determine the horizontal displacements of the structure under the applied loads. Next, a fictitious horizontal load (H_f) proportional to the product of the axial load (P) and the lateral displacement obtained (Δ) is calculated, representing the increment of moment due to the horizontal displacement of the structure. This fictitious load is then applied to the structure, and a new analysis is performed, obtaining new displacements (Δ_i). The

process is iteratively repeated until the displacements and internal forces converge within a pre-defined tolerance criterion.

The first-order displacement Δ_1 can be obtained by known Mechanics of Materials methods such as the Principle of Virtual Works and the Elastic Curve Equation, using the secant stiffness (EI_{sec}), in order to take into account the physical nonlinearity of the elements. The bending moment at the base of the column, considering the 1st order analysis with the addition of 2nd order effects, is expressed by:

$$M_{2,d} = M_{1d} + \mathrm{P}\Delta_1 \tag{3}$$

where $M_{1d} = HL$ is the 1st-order moment generated by the load H and $P\Delta_1$ represents the additional second-order moment generated by the axial force P acting on the lateral displacement Δ.

From this, the fictitious side load is introduced to the ith iteration, which simulates the effects of the 2nd order:H_{fi}

$$H_{fi} = \frac{\mathrm{P}\Delta_i}{L}. \tag{4}$$

With this dummy load, the updated calculation time of the following iteration is recalculated:

$$M_{i,d} = M_{1,d} + \mathrm{P}\Delta_{i+1} = HL + H_{fi}\mathrm{L} = \big(\mathrm{H} + H_{fi}\big)L. \tag{5}$$

At the end of the iterative process, the total time of calculation, considering the n iterations, is given by:

$$M_{tot,d} = M_{1,d} + \sum_{i=1}^{n} P\Delta_{i+1}. \tag{6}$$

With this moment, the structural element is designed.

The convergence criterion of the iterative process can be defined from a previously defined tolerance ξ, so that the iteration will be interrupted when the relative difference between the consecutive calculation moments becomes sufficiently small. This criterion can be expressed mathematically by:

$$\xi_i = \frac{M_{i,d}}{M_{i-1,d}} \leq \xi. \tag{7}$$

2.2 Equivalent stiffiness $\left(EI_{eq}\right)$ and Simpson's Rule

The use of slender structural elements with variable cross-section is quite common in civil construction, since there is a consolidated trend in the sector to reduce the weight of the structures and, consequently, the costs of the work. In addition, the advancement of structural engineering has provided architects with greater freedom to dare in their projects, incorporating this type of element more recurrently.

The variation of the cross-section along the length of an element implies non-constant stiffness, which directly impacts its structural behavior and design. Several analytical and numerical methods allow this variation to be considered throughout the analysis of the structure. In this research, we propose the application of a simple but effective method, capable of analyzing structural elements with variable cross-section — whether these sections are usual or unusual. The equivalent stiffness is determined as follows:

$$(EI)_{eq} = \frac{1}{L}\int_{x_0}^{x_n} (EI)dx = \frac{E}{L}\int_{x_0}^{x_n} Idx \tag{8}$$

where L is the length of the structural element, E is the longitudinal modulus of elasticity of the material, and I is the moment of inertia of the bending axis.

Simple Simpson's Rule or First Simpson's Rule aims to approximate a function $f(x)$ by a second-degree Lagrange interpolating polynomial, represented by $P_2(x)$. For this, three known points are needed, $x_0 = a, x_1\ e\ x_2 = b$, equally spaced from h.

$$P_2(x) = \mathrm{L}_0(x)\ \mathrm{f}(x_0) + \mathrm{L}_1(x)\ \mathrm{f}(x_1) + \mathrm{L}_2(x)\ \mathrm{f}(x_2), \tag{9}$$

being

$$L_i(x) = \prod_{\substack{j=0 \\ i\neq j}}^{n} \frac{(x-x_j)}{(x_i-x_j)},\ \text{i=0,1,2.} \tag{10}$$

Thus, it is possible to admit that:

$$\int_a^b f(x)dx \cong \int_a^b P_2(x)dx. \tag{11}$$

Expanding the integral, we have:

$$\int_a^b f(x)dx \cong \int_a^b \frac{(x-x_1)}{(x_0-x_1)}\frac{(x-x_2)}{(x_0-x_2)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}\frac{(x-x_2)}{(x_1-x_2)}f(x_1) + \frac{(x-x_0)}{(x_2-x_0)}\frac{(x-x_1)}{(x_2-x_1)}f(x_2) \tag{12}$$

Adopting $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$, and resolving the integral, we arrive at:

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \tag{13}$$

For more points, just use the formulas over and over again. For example, for five points, the integral would be:

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] + \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)]. \tag{14}$$

Soon

$$\int_a^b f(x)dx \cong \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \tag{15}$$

It should be noted that integrals can be solved numerically by the Gauss-Legendre method, in this case, knowing that:

$$\int_{x_0}^{x_2} f(x)dx \cong \sum_{i=1}^{n} f(x_i)w_i, \tag{16}$$

being n the number of Gauss points and W_i the Gauss weights, we have:

$$\int_{x_0}^{x_2} f(x)dx = f(x_0)w_0 + f(x_1)w_1 + f(x_2)w_2 \tag{17}$$

where, $w_0 = \int_{x_0}^{x_2} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx = \frac{h}{3}$; $w_1 = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}dx = \frac{4h}{3}$; $w_2 = \int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}dx = \frac{h}{3}$.

Finally, through Simpson's Rule, it is possible to determine the equivalent stiffness of the element, whose section is variable through eq. (18).

$$(EI)_{eq} = \frac{E}{L}\int_{x_0}^{x_n} Idx = \frac{E}{L}\sum_{i=1}^{n} I_i W_i. \tag{18}$$

3 Methodology

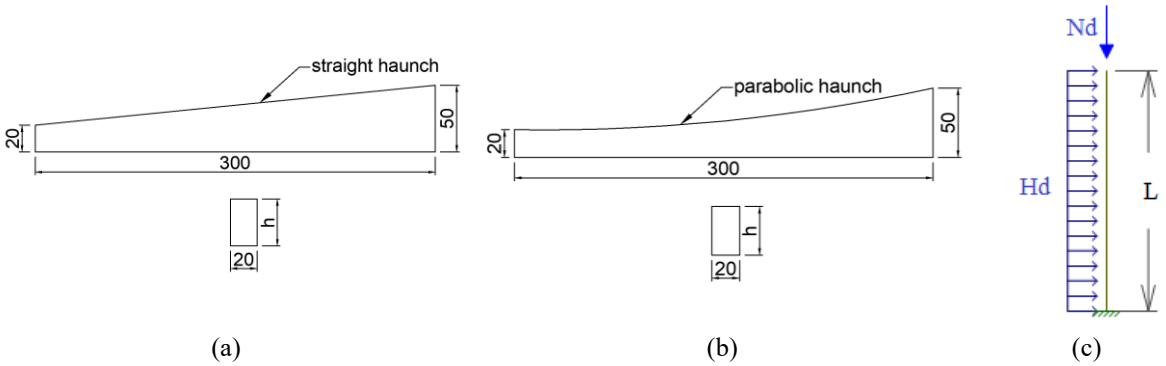
This research proposes the simplified geometric and physical nonlinear analysis of slender columns with constant cross-section and haunch variation. To calculate the equivalent stiffness of the columns, Simpson's Rule was

adopted, while the physical nonlinearity of the material was considered based on the approximate values of secant stiffness recommended by the Brazilian standard, using $EI_{sec} = 0{,}8(EI)_{eq}$. To consider the global effects of 2nd order, a computational algorithm based on the iterative process P-Δ was developed, with the application of a fictitious lateral load. The results achieved in this analysis were then compared with those obtained by the method of amplification of efforts with the coefficient γ_z.

4 Applications

4.1 Application 1

Perform the global 2nd order analysis of the structural elements of Figure 1, that is, determine the total 2nd order moment of the reinforced concrete column, $fck = 20MPa$, fixed-base cantilever column subjected to the following design loading: centered compression load N_d and evenly distributed H_d horizontal load. Table 1 presents the results obtained for cases (a) and (b) in Figure 1.



(a) (b) (c)

Figure 1. (a) Straight *haunch*, number of sections $n = 11$, tolerance $\xi = 10^{-4}$, loads $N_d = 1500kN$, $H_d = 12kN/m$ and $L = 3m$; (b) Parabolic *haunch*, number of sections $n = 11$, tolerance $\xi = 10^{-4}$, loads $N_d = 1500kN$, $H_d = 12kN/m$ and $L = 2{,}5m$; (c) Fixed-base cantilever column.

Table 1. 2nd order analysis of the cantilever column considering haunch variation

Case	$(EI)_{eq}$ (Nm²)	$(EI)_{sec}$ (Nm²)	$M_{d,tot}$ (iterative) (kNm)	(γ_z)	$M_{d,tot}$ (γ_z)c (kNm)	Iterations
a	2.118×10^7	1.695×10^7	68.64	1.249	64.051	6
b	1.45×10^7	1.16×10^7	47.902	1.254	44.67	6

4.2 Application 2

A similar cantilever colunm to the previous application is considered, but with a constant square cross-section of width b, lengh $L = 3{,}5m$ and fck = 25 MPa, subjected to the loading $H_d = 12$ kN/m and N_d. The dimensionless bending moment μ, was evaluated by the numerical method and the coefficient γ_z method, for different values of b and N_d. Table 2 presents the results obtained.

Table 2. Dimensionless 2nd order moment of the fixed-base cantilever column and constant square section

Section (cm)	N_d (kN)	μ_{num}	μγz
29x29	1000	0.225	0.2197
30x30	1100	0.20	0.196
31x31	1200	0.1793	0.1756

33x33	1300	0.14	0.1396
34x34	1400	0.128	0.1264
35x35	1500	0.1161	0.1148
36x36	1600	0.1056	0.1045

5 Conclusions

Based on the analyses and results presented in the present work, it is verified that the application of Simpson's Rule for the determination of the equivalent stiffness of columns with variable cross-section, associated with the iterative method P-Δ with fictitious lateral loading, proved to be efficient and compatible with the normative precepts of ABNT NBR 6118:2023 [1]. The methodology allowed to incorporate the effects of geometric nonlinearity and also the effects of physical nonlinearity in a simplified way, and the results were slightly more conservative than the answers proposed with the aid of the normative method based on the coefficient γ_z, especially in cases of greater slenderness or sections more susceptible to loss of rigidity.

The comparison between the 2nd-order bending moments obtained by the iterative process P-Δ and those estimated by amplification with the coefficient γ_z (which is also an iterative process) demonstrated coherence, but also evidenced the potential of the proposed method to capture more refined effects of global instability. In this way, the developed algorithm contributes to a more realistic and safe analysis of slender columns in reinforced concrete, being especially useful in contexts where the variation of the section along the element significantly impacts the structural behavior.

Acknowledgments. Gratefully acknowledges financial support from the Foundation Coordination for the Improvement of Higher Education Personnel (CAPES) and of the Postgraduate Program in Civil Engineering at the Federal University of Ouro Preto.

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