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Nonlinear Analysis of Prismatic Reinforced Concrete Structures under Harmonic Loads

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Abstract. Dynamic analysis of structures differs from static analysis primarily in its consideration of time-dependent load variations. Harmonic loads, or cyclic loads, are among the most critical, requiring accurate evaluation to predict structural behavior. These loads vary periodically and can cause complex responses, including nonlinear effects. Therefore, structural design must ensure safety and comfort, avoiding unexpected performances. Computational simulations are essential in dynamic analysis, providing precise results and allowing the assessment of complex behaviors. This ensures that both static (self-weight, permanent loads) and dynamic actions (wind, vibrations, impacts) are properly considered. The main objective of this work was to develop a methodology for the dynamic analysis of 2D reinforced concrete structures, considering the material nonlinearity (cracking). The Finite Element Method (FEM) was used for spatial discretization, and the Newmark method for temporal discretization. Material nonlinearity was modeled using the constitutive equation from ABNT NBR 6118:2023, with time-dependent updates to the stiffness matrix. The Newton-Raphson method was applied to solve the nonlinear problem during reinforced concrete section verification. Results were compared with those from established software (student, free, or trial versions) to validate the proposed algorithm's robustness and efficiency.

Keywords: Dynamic Analysis, Reinforced concrete, Nonlinearity, Finite Element Method (FEM), Harmonic loads.

1 Introduction

The nonlinear analysis of planar reinforced concrete frames has been a subject of study for many years [1]. Several methodologies have been developed, primarily for structures subjected to static loads. However, reinforced concrete, being a versatile and easily moldable material, is increasingly used in infrastructure projects. These structures are subjected to various types of loads during their service life, one of which is dynamic loading. Therefore, understanding the resistance mechanisms of this material under time-dependent loads is crucial, especially since reinforced concrete exhibits cracking and, consequently, a loss of stiffness at typical or service load levels.

Currently, some studies present methodologies aimed at analyzing the mechanical behavior of concrete structures subjected to dynamic loads (see Rosero [2] and Rogério [3]). These studies use the Finite Element Method (FEM) (see Bathe [4]) as a numerical method to obtain an approximate solution to the problem. According to Rosero [2], the main challenges lie in modeling the reinforcement, including issues such as the placement of steel elements, the lack of understanding of the mechanical behavior of the steel bars, and their interaction with the concrete. These challenges in reinforcement modeling arise from assumptions such as perfect bond between steel and concrete or the superposition of perfectly distributed steel within the concrete elements, treating reinforced concrete as a homogeneous material.

Thus, this work aims to contribute to the field of structural analysis by examining the dynamic behavior of prismatic bar structures made of reinforced concrete, considering the material nonlinearity of the constituent materials. MATLAB platform was used to develop a program that discretizes the structure under study and obtains

dynamic responses to harmonic loads.

2 The Dynamic Problem

Considering a single-degree-of-freedom oscillator, including a linear viscous damper equivalent to energy dissipation and subjected to harmonic loading, as described by Clough and Penzien [5], the general equation of motion can be defined as

$$m \cdot \ddot{u}(t) + c \cdot \dot{u}(t) + k \cdot u(t) = p_0 \operatorname{sen}(\bar{\omega} t) \tag{1}$$

where m, c and k are the mass, damping, and stiffness constants, respectively. The terms $\ddot{u}(t)$, $\dot{u}(t)$ and $u(t)$ represent acceleration, velocity, and displacement, respectively, all as functions of time t. In the eq. (1), due to harmonic loading, there is a stationary component, where p_0 is the amplitude and $\bar{\omega}$ is the loading frequency.

By discretizing the mass, damping, and stiffness using the Finite Element Method (FEM), we can rewrite the eq. (1) as follows

$$\mathbf{M} \cdot \ddot{\mathbf{u}}(t) + \mathbf{C} \cdot \dot{\mathbf{u}}(t) + \mathbf{K} \cdot \mathbf{u}(t) = \mathbf{r} \tag{2}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping, and stiffness matrices, respectively, and $\ddot{\mathbf{u}}(t)$, $\dot{\mathbf{u}}(t)$, $\mathbf{u}(t)$ and $\mathbf{r}$ are the acceleration, velocity, displacement, and time-varying force vectors, respectively.

2.1 Time and spatial discretization

In temporal discretization, the time variable is divided into time instants separated by intervals Δt, referred to as the integration step. In a generalized form, the Newmark method (see Newmark [6]) can be expressed as follows

$$\hat{\mathbf{K}}\mathbf{u}_{i+1} = \hat{\mathbf{r}}_{i+1} \tag{3}$$

where

$$\hat{\mathbf{K}} = \mathbf{K} + \frac{1}{\alpha \Delta t^2}\mathbf{M} + \frac{\delta}{\alpha \Delta t}\mathbf{C} \tag{4}$$

$$\begin{aligned} \hat{\mathbf{r}}_{i+1} = \mathbf{r}_{i+1} &+ \mathbf{M}\left[\frac{1}{\alpha \Delta t^2}\mathbf{u}_i + \frac{1}{\alpha \Delta t}\dot{\mathbf{u}}_i + \left(\frac{1}{2\alpha} - 1\right)\ddot{\mathbf{u}}_i\right] \\ &+ \mathbf{C}\left[\frac{\delta}{\alpha \Delta t}\mathbf{u}_i + \left(\frac{\delta}{\alpha} - 1\right)\dot{\mathbf{u}}_i + \left(\frac{\delta}{\alpha} - 2\right)\frac{\Delta t}{2}\ddot{\mathbf{u}}_i\right] \end{aligned} \tag{5}$$

The values associated with the subscript i are calculated at time $t = i\Delta t$, where Δt is the constant time increment used for the integration step. The constants δ and α are the Newmark's integration parameters and they must satisfy the following conditions to ensure the method is unconditionally stable:

$$\delta \geq \frac{1}{2} \text{ and } \alpha \geq \frac{1}{4}\left(\frac{1}{2} + \delta\right)^2 \tag{6}$$

The eq. (7) and (8) can be used to determine the acceleration and velocity vectors at time step $i + 1$ in the eq. (5).

$$\ddot{\mathbf{u}}_{i+1} = \frac{1}{\alpha \Delta t^2}\left(\mathbf{u}_{i+1} - \mathbf{u}_i\right) - \frac{1}{\alpha \Delta t}\dot{\mathbf{u}}_i - \left(\frac{1}{2\alpha} - 1\right)\ddot{\mathbf{u}}_i \tag{7}$$

$$\dot{\mathbf{u}}_{i+1} = \dot{\mathbf{u}}_i + (1 - \delta)\Delta t\,\ddot{\mathbf{u}}_i + \delta \Delta t\,\ddot{\mathbf{u}}_{i+1} \tag{8}$$

3 The Nonlinear Problem

Consider a reinforced concrete element with a generic cross-section subjected to the loads, N_z, M_x and M_y, as shown in Fig. 1.

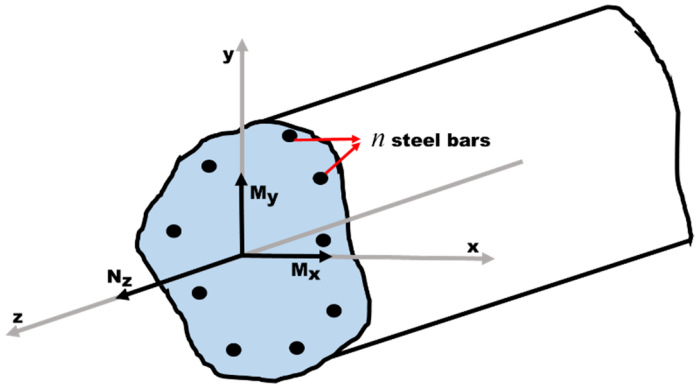


Figure 1: Reinforced concrete beam of generic section subjected to normal load and moments about the x and y axes.

Using a sign convention such that moments and curvatures are consistent, the normal force and bending moments, both resistant, are given by:

$$\begin{aligned} N_z &= \int_A \sigma_z^c \, dA + \sum_{i=1}^{n} \sigma_z^{s,i} A_{s,i}, \; M_x = \int_A y \sigma_z^c \, dA + \sum_{i=1}^{n} y_i \sigma_z^{s,i} A_{s,i} \text{ and} \\ M_y &= -\left(\int_A x \sigma_z^c \, dA + \sum_{i=1}^{n} x_i \sigma_z^{s,i} A_{s,i} \right) \end{aligned} \tag{9}$$

where x and y are the coordinates of the infinitesimal dA of the concrete element in the plane xy and σ_z^c is the normal stress in this same infinitesimal element of the concrete. n is the numbers of steel bars, x_i and y_i are the coordinates in the plane xy of the steel bars, $A_{s,i}$ is the cross-sectional area of bar i and $\sigma_z^{s,i}$ is the normal stress of the same bar.

Based on the Euler-Navier-Bernoulli hypotheses, i.e., the plane section before loading remains plane after the application of the loading, the eq. (10) can be used as the strain (ε) of any point in the cross-section.

$$\varepsilon = \varepsilon_0 - k_y x + k_x y \tag{10}$$

where ε_0 is the strain at the origin of the coordinate system; k_x and k_y are the curvatures of the section in the x and y directions, respectively.

To verify whether a section with determined geometry and reinforcement resists the solicitant efforts, considered known, the Newton-Raphson method is used to state the following problem: determine $\mathbf{x}$, or rather, solve $\mathbf{f} = \mathbf{r}_{ext} - \mathbf{r}_{\text{int}} = 0$, knowing the solicitant efforts $N_{z,d}$, $M_{x,d}$ and $M_{y,d}$, the geometry of the section, and the arrangement of the reinforcement, where:

$$\mathbf{x} = \begin{Bmatrix} \varepsilon_0 \\ k_y \\ k_x \end{Bmatrix}, \; \mathbf{r}_{ext} = \begin{Bmatrix} N_{z,d} \\ M_{x,d} \\ M_{y,d} \end{Bmatrix}, \; \mathbf{r}_{\text{int}} = \begin{Bmatrix} N_z \\ M_x \\ M_y \end{Bmatrix} \text{ and } \mathbf{f} = \begin{Bmatrix} N_{z,d} - N_z \\ M_{x,d} - M_x \\ M_{y,d} - M_y \end{Bmatrix} \tag{11}$$

To obtain a solution to the exposed problem, the Newton-Raphson method is used. For this, it is necessary to determine the Jacobian matrix of $\mathbf{f}$, where $\mathbf{K}^t = \partial \mathbf{f} / \partial \mathbf{x} = \nabla \mathbf{f}$. Thus:

$$\mathbf{K}^t = \begin{bmatrix} -\int_A D^c\,dA - \sum_{i=1}^{n} D_i^s A_i^s & -\int_A D^c\,y\,dA - \sum_{i=1}^{n} D_i^s y_i^s A_i^s & \int_A D^c\,x\,dA + \sum_{i=1}^{n} D_i^s x_i^s A_i^s \\ -\int_A D^c\,y\,dA - \sum_{i=1}^{n} D_i^s y_i^s A_i^s & -\int_A D^c\,y^2\,dA - \sum_{i=1}^{n} D_i^s (y_i^s)^2 A_i^s & \int_A D^c\,xy\,dA + \sum_{i=1}^{n} D_i^s x_i^s y_i^s A_i^s \\ \int_A D^c\,x\,dA + \sum_{i=1}^{n} D_i^s x_i^s A_i^s & \int_A D^c\,xy\,dA + \sum_{i=1}^{n} D_i^s x_i^s y_i^s A_i^s & -\int_A D^c\,x^2\,dA - \sum_{i=1}^{n} D_i^s (x_i^s)^2 A_i^s \end{bmatrix} \tag{12}$$

where $D^c = \partial\sigma_z^c / \partial\varepsilon_z^c$ and $D^s = \partial\sigma_z^{s,i} / \partial\varepsilon_z^{s,i}$; ε_z^c and $\varepsilon_z^{s,i}$ are the strains of the concrete and steel bar i, respectively. In the Ultimate Limit State (ULS), the approximate stress-strain diagrams for concrete and steel used in this work are the same as suggested by ABNT NBR 6118:2023 [7], as shown in Fig. 2.

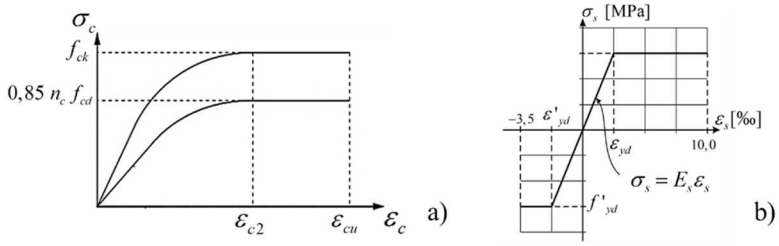


Figure 2: (a) Stress-strain diagram for concrete (ABNT NBR 6118:2023); (b) Stress-strain diagram for steel (ABNT NBR 6118:2023).

Thus, D^c and D_i^s, become:

$$D^c = \begin{cases} 0 \text{ to } \varepsilon_z^c \geq 0 \\ \dfrac{0{,}85 n_c f_{cd}\, n}{\varepsilon_{c2}} \left(1 - \dfrac{\varepsilon_c}{\varepsilon_{c2}}\right)^{n-1} \text{ to } 0 > \varepsilon_z^c > \varepsilon_{c2} \\ 0 \text{ to } \varepsilon_z^c \leq \varepsilon_{c2} \end{cases}$$

$$D_i^s = \begin{cases} 0 \text{ to } \left|\varepsilon_z^{s,i}\right| \geq \varepsilon_{yd} \\ E^s = \dfrac{f_{yd}}{\varepsilon_{yd}} \text{ to } \left|\varepsilon_z^{s,i}\right| < \varepsilon_{yd} \end{cases} \tag{13}$$

where ε_{c2} and ε_{yd} are the design elastic strain limits for concrete and steel, respectively. It is assumed, as a hypothesis, that the concrete fills the entire section, allowing division into N rectangular cells, which also makes the eq. (12) discrete with respect to the concrete portion. Therefore, to solve the nonlinear problem of eq. (11), the following iterative Newton-Raphson procedure is performed:

$$\begin{aligned} &\text{Initial estimate } \rightarrow \mathbf{x}_0 \\ &\text{Solve } \rightarrow \mathbf{K}^t\big|_{\mathbf{x}=\mathbf{x}_n} \cdot \boldsymbol{\delta} = -\mathbf{f}(\mathbf{x}_n) \\ &\text{Update } \rightarrow \mathbf{x}_{n+1} = \mathbf{x}_n + \boldsymbol{\delta} \\ &\begin{cases} \text{If } \left|\mathbf{f}(\mathbf{x}_n)\right| < tol \rightarrow \text{solution obtained} \\ \text{If } \left|\mathbf{f}(\mathbf{x}_n)\right| > tol \rightarrow n = n + 1 \rightarrow \text{restart the loop} \end{cases} \end{aligned} \tag{14}$$

4 Nonlinear Dynamic Analysis of reinforced concrete elements

The dynamic analysis of reinforced concrete structures, considering material nonlinearity, requires combining numerical time integration methods (Newmark) with methods for solving nonlinear problems (Newton-Raphson). The methodology of the adopted algorithm is summarized in Fig. 3.

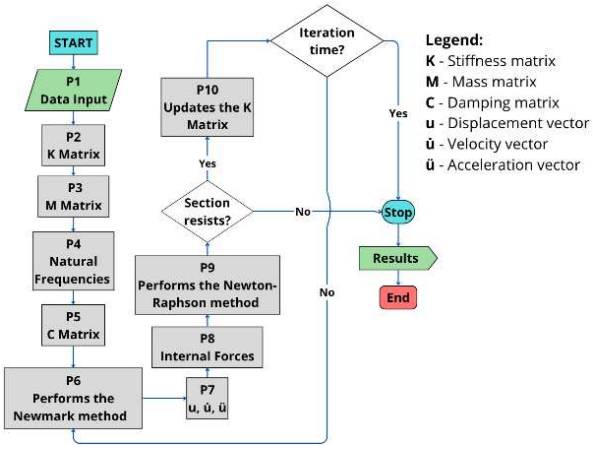


Figure 3: Flowchart for Nonlinear dynamic analysis.

The algorithm begins at Step 1 (P1), with the reading of the input data file containing all the information necessary for the analysis. At Step 2 (P2), the stiffness matrix of each element is calculated, and the global stiffness matrix of the structure is assembled, followed by the application of boundary conditions. Step 3 (P3) is similar to the previous one but refers to the assembly of the global mass matrix of the structure. At Step 4 (P4), the matrix with the natural frequencies and the modal matrix are obtained. Step 5 (P5) consists of obtaining the structure's damping matrix through a linear combination between $\mathbf{K}$ and $\mathbf{M}$ (Rayleigh-type damping). At Step 6 (P6), the temporal integration of the equation of motion begins via the Newmark method. At Step 7 (P7), the displacement, velocity, and acceleration vectors are obtained for a given time interval Δt. Step 8 (P8) deals with obtaining the internal forces of the structure, which are used in Step 9 (P9) to verify all sections of the structure using the Newton-Raphson method.

If there is no convergence in P9, the process is terminated, indicating that the section defined by the user in P1 does not resist the solicitant efforts. If convergence occurs, the process proceeds with the update of the stiffness matrix $\mathbf{K}$ (P10), based on the tangent stiffness matrix ($\mathbf{K}^t$) of each verified section. For each element, two matrices are obtained: one for the section at the initial node (IS) and another for the section at the final node (FS). The averages of the axial stiffness (EA) and flexural stiffness (EI) are calculated, thus updating the local stiffness matrix of each element. After this, the global matrix $\mathbf{K}$ is reassembled, applying the boundary conditions. If the total time t, stablished by the user in P1, is reached, the process ends, providing a valid solution for the problem in question. Otherwise, the Newmark method (P6) restarts with the time t updated by $t = t + \Delta t$.

5 Undamped Planar Frame

To exemplify the methodology presented in the previous section, this example consists of discretizing an undamped planar frame, considering mass only along the beam's length. The planar frame, with its respective finite element discretization, and the beam and column sections, are presented in Fig. 4 and Fig. 5.

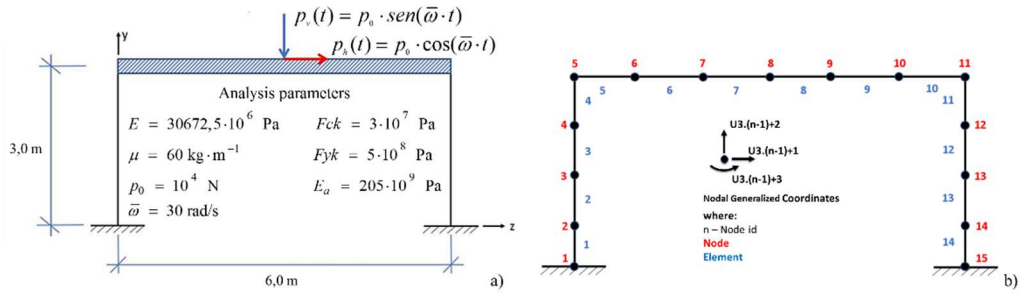


Figure 4: (a) Undamped planar frame; (b) Finite element discretization.

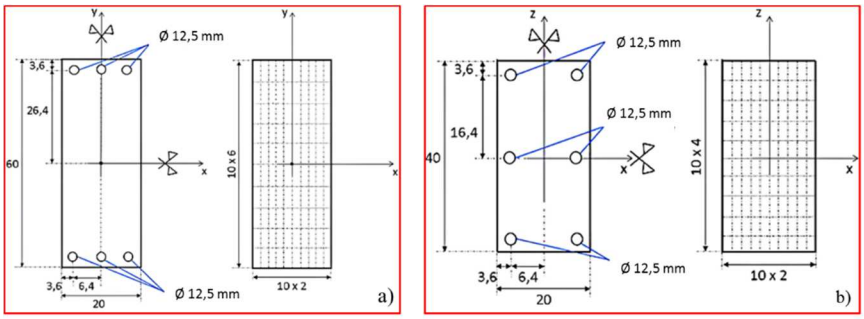


Figure 5: (a) Column section with discretization mesh (measures in cm); (b) Beam section with discretization mesh (measures in cm).

For the verification of the developed program's algorithm, the frame was modeled in SAP2000 and Ansys, and a comparison was made between the first five linear natural vibration frequencies of the structure, confirming the convergence of the results, as presented in Fig. 6(a). Additionally, the frame was verified for the loading frequency ($\bar{\omega}$) indicated in Fig 4(a), as well as for specific frequencies chosen to study the dynamic responses, referring to node 8 (located at the mid-span of the beam) of the planar frame in question, as shown in Fig. 6(b). The ω_1 and ω_2 are the nonlinear natural frequencies of the first and second vibration modes, respectively.

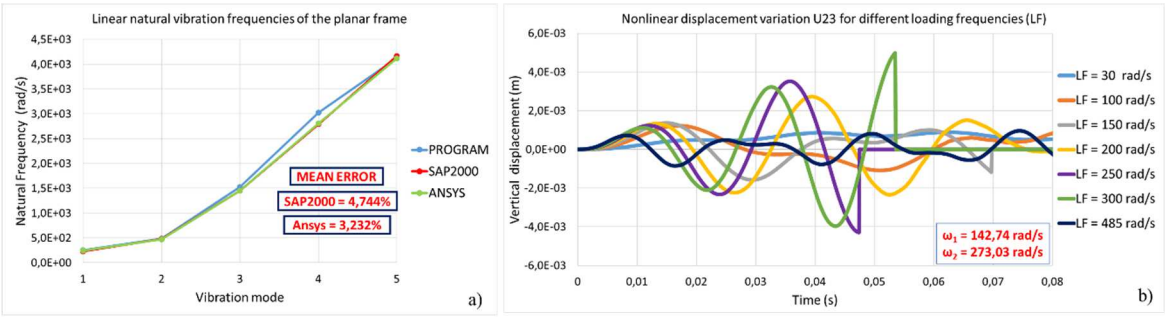


Figure 6: (a) Linear Natural Vibration Frequencies of the planar frame; (b) Loading frequencies at Node 8.

Analyzing Fig 6(b), it is observed that for loading frequencies (LF) of 150 rad/s, 250 rad/s, and 300 rad/s, the frame reaches the Ultimate Limit State (ULS). Considering the vertical displacement at node 8, the results may seem contradictory at first look: for the first frequency mentioned, the displacement associated with failure is relatively small compared to the rupture displacement observed for frequencies of 250 rad/s and 300 rad/s. This difference is explained by analyzing where the failure occurs in the structure. The frame has distinct reinforced concrete sections in both the columns and the beam. For a frequency of 150 rad/s, the failure occurs at the base of the column, at node 1, while for frequencies of 250 rad/s and 300 rad/s, the rupture occurs in the beam, more precisely at node 8. These conclusions are based on the analysis of the solicitant efforts, presented in Fig. 7(a) and Fig. 7(b), which show the results obtained for frequencies of 150 rad/s and 250 rad/s.

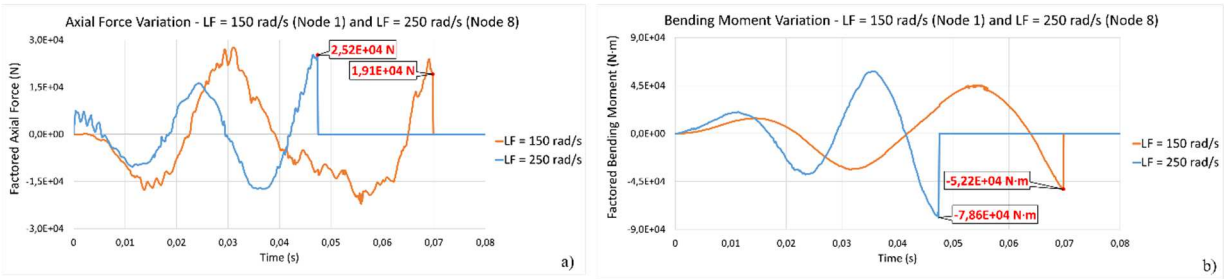


Figure 7: (a) Axial force history; (b) Bending moment history.

Based on the interaction diagrams in Fig. 8, it is verified that the rupture of the reinforced concrete sections occurred due to the steel's strain limit being exceeded, both in the column and the beam, for each loading frequency. Resonance at the analyzed LF is expected, as $\beta_1 = 150$ rad.s$^{-1}/\omega_1 \cong 1$ and $\beta_2 = 250$ rad.s$^{-1}/\omega_2 \cong 1$.

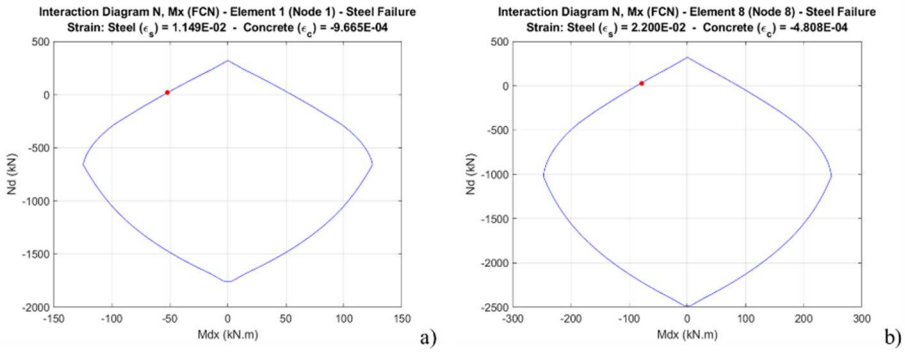


Figure 8: (a) Interaction Diagram - LF = 150 rad/s; (b) Interaction Diagram - LF = 250 rad/s.

6 Conclusions

This work presented a methodology to simulate 2D unidimensional reinforced concrete structures subjected to harmonic loads. Structural discretization was performed using the FEM, with temporal integration by the Newmark method, combined with the Newton-Raphson method to consider the material nonlinearity of the constituent materials of the reinforced concrete section. The results were validated with the aid of the widely used SAP2000 and Ansys software (both trial versions). Due to space limitations, the modeling details in these programs are not presented in this work. The developed program demonstrated the ability to accurately identify the critical loading frequencies that lead to section failure, as well as the structural element, node, solicitant effort responsible for the rupture, and the material that exceeded the strain limit, according to the criteria of ABNT NBR 6118:2023 [7].

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