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Optimal Design with Connectivity Constraints via Level-Set and Topological Derivatives Considering an Auxiliary Eigenvalue Problem

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Abstract. This work addresses the issue of imposing connectivity constraints in shape and topology optimization, with the goal of controlling the presence of void or solid islands in optimal designs. The approach relies on the spectrum of a two-phase differential operator with Dirichlet boundary conditions imposed on the external boundary. In contrast to previous works that adopt the same mathematical formulation for connectivity within the framework of density-based methods, we consider a level set representation of the domain combined with a topological-derivative-based optimization algorithm. Numerical experiments are presented to validate the strategy.

Keywords: Connectivity constraints, topological derivatives, level-set method, eigenvalue constraints.

1 Introduction

Recent advances in topology optimization (TO) have enabled the design of highly complex and efficient structures. While the development of additive manufacturing (AM) technologies has played a critical role in ensuring that optimal solutions are in fact manufacturable, it has also highlighted the need to incorporate additional *geometrical constraints* into topology optimization formulations. In this work we focus on the issue of imposing *connectivity constraints,* aiming to prevent the existence of isolated regions of solid or void material in optimal designs (see Figure 1). From a 3D printing perspective, *void islands*, also known as *enclosed* or *internal holes*, pose important challenges: they can result in the encapsulation of residual material (in powder or resin-based techniques, for instance) and often require internal supports that cannot be removed, leading to undesired permanent features.

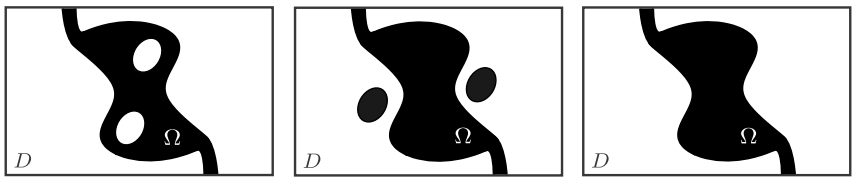


Figure 1. Domain with void islands (left), with solid islands (middle) and without solid or void islands (right).

Different strategies have been proposed over the years to enforce connectivity constraints in topology optimization (for a complete review, see [1] and references therein). The virtual temperature method, introduced by Li et al. [2], is a physics-based approach that ensures connectivity by creating artificial heat diffusion and imposing temperature constraints. Alternatively, geometry-based methods have also emerged, such as the spectral graph approach of Donoso and Guest [3]. This discrete method was later extended by the same authors to the continuous setting using the spectrum of the Neumann-Laplace operator [4] and further using the Dirichlet-Laplace operator (see Donoso et al. [5]), which is the formulation adopted in this work. Unlike the previously mentioned works,

however, that adopt density-based formulations for the topology optimization setting, we consider a level-set representation of the domain combined with a topological derivative-based algorithm, combining elements from [6–8].

2 Problem Formulation

Let us consider a piecewise smooth, open and bounded set $\mathcal{D} \subset \mathbb{R}^2$. For the present application, $\mathcal{D}$ represents a fixed domain which is split into two subdomains with different material properties: Ω and its complement $\mathcal{D}\backslash\Omega$. For each set $\Omega \subset \mathcal{D}$, let us define the characteristic function $\chi_\Omega \in X \coloneqq L^\infty(\mathcal{D}; \{0,1\})$ as

$$\chi_\Omega(x) \coloneqq \begin{cases} 1, & x \in \Omega, \\ 0, & x \in \mathcal{D}\backslash\Omega. \end{cases} \tag{1}$$

As a model problem to investigate connectivity constraints, we will consider the classic problem of minimizing the structural compliance of the domain under volume constraints, with the additional requirement that the optimal configuration has no *enclosed holes*. The topology optimization formulation is introduced in Subsection 2.1. Then, the mathematical model to identify and avoid enclosed holes is described in Subsection 2.2.

2.1 A Model Problem in Linear Elasticity

Let us assume that $\partial\mathcal{D} = \Gamma_d \cup \Gamma_n$ and define $H^1_d(\mathcal{D}) \coloneqq \{u \in H^1(\mathcal{D}) : u = 0 \text{ on } \Gamma_d\}$. Also, let $\mathcal{O} \coloneqq \{\Omega \subset \mathcal{D} \text{ such that } \Omega \text{ has no enclosed voids}\}$ denote the set of admissible domains, in terms of connectivity. We are interested in the following shape and topology optimization problem

$$\min_{\chi_\Omega \in X} \quad J(u(\chi_\Omega)), \quad \text{subject to} \quad |\Omega| = V^* \quad \text{and} \quad \Omega \in \mathcal{O}, \tag{2}$$

where the compliance shape functional is defined as

$$J(u(\chi_\Omega)) = \int_{\mathcal{D}} \nabla^s u : \mathbb{C}(\chi_\Omega) : \nabla^s u, \tag{3}$$

and the displacement $u \in (H^1_d(\mathcal{D}))^n$ is the solution of

$$\int_{\mathcal{D}} \nabla^s u : \mathbb{C}(\chi_\Omega) : \nabla^s v = \int_{\Gamma_n} gv, \quad \forall\, v \in (H^1_d(\mathcal{D}))^n, \tag{4}$$

with $\nabla^s u \coloneqq (\nabla u + \nabla u^\top)/2$ and $g \in L^2(\Gamma_n)$ a given prescribed force. The material properties are described by the fourth-order tensor $\mathbb{C}(\chi_\Omega) \coloneqq \mathbb{C}_1\chi_\Omega + \mathbb{C}_0(1-\chi_\Omega)$. For plane stress linear elasticity we have $\mathbb{C}_1 \coloneqq 2\mu_1\mathbb{I} + (\kappa_1 - \mu_1)\mathrm{I}\otimes\mathrm{I}$ and $\mathbb{C}_0 \coloneqq 2\mu_0\mathbb{I} + (\kappa_0 - \mu_0)\mathrm{I}\otimes\mathrm{I}$, where $\mathbb{I}$ and I are the fourth and second order identity tensors and κ_1, κ_0 and μ_1, μ_0 are the bulk and shear modulus of phase Ω and $\mathcal{D}\backslash\Omega$, respectively.

2.2 An Auxiliary Eigenvalue Problem to Identify Enclosed Voids

Let us recall that, in theory of sets, an arbitrary domain $\omega \subset \mathbb{R}^n$ is said to be connected if it cannot be partitioned into two open, non-empty and disjoint sets. Furthermore, if a set is not connected, then it can be expressed as the disjoint union of connected sets. In fact, let us assume that the void phase $\mathcal{D}\backslash\Omega$ can be expressed as the disjoint union of the connected subsets $\omega_1, \ldots, \omega_m$, with $m \geq 1$. Then, a void island can be defined as a set ω_i that is disconnected from the external boundary, as illustrated in Figure 2. The eigenvalue-based approach that we adopt consists in using the spectrum of an operator to identify subsets with these properties.

Consider the following two-phase eigenvalue problem with Dirichlet boundary condition imposed on $\partial\mathcal{D}$,

$$\text{find } \phi \in H^1_0(\mathcal{D}) : \int_{\mathcal{D}} \alpha(\chi_\Omega)\nabla\phi \cdot \nabla v dx = \lambda \int_{\mathcal{D}} w(\chi_\Omega)\phi v, \quad \forall v \in H^1_0(\mathcal{D}), \tag{5}$$

where $\alpha(\chi_\Omega) \coloneqq \alpha_0\chi_\Omega + (1-\chi_\Omega)$ and $w(\chi_\Omega) \coloneqq w_0\chi_\Omega + (1-\chi_\Omega)$, with $0 < \alpha_0, w_0 \ll 1$, and $\lambda \in \mathbb{R}$ denotes the eigenvalue associated with the eigenfunction $\phi \in H^1_0(\mathcal{D})$. Note that, in contrast with the elastic problem, the small material properties have now been assigned to phase Ω. We can then verify that in the limit as α_0, w_0 tend to zero, we approximate the eigenvalues of the Laplace operator posed on $\mathcal{D}\backslash\Omega$, with Dirichlet boundary conditions on the portions of the boundary that intersect $\partial\mathcal{D}$ and homogeneous Neumann conditions otherwise. Two key results

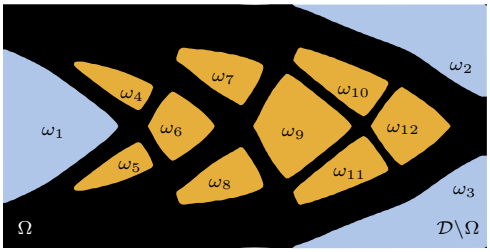


Figure 2. *Enclosed holes* in a typical solution of a minimum compliance problem with volume constraints (without connectivity constraints). We have $\mathcal{D}\backslash\Omega = \sum_{i=1}^{12} \omega_i$ and the yellow sets are identified as internal voids.

regarding spectral properties of the Laplace operator support our formulation. First, the smallest eigenvalue of the Neumann-Laplace operator on a connected set is zero, whereas if Dirichlet boundary conditions are imposed on a not empty portion of the boundary, the first eigenvalue becomes strictly positive. Second, if the problem is posed on a disconnected domain, the eigenvalues can be obtained by solving the problem on each connected component. Combining these elements, one can ensure connectivity of the domain by imposing that the first eigenvalue is strictly positive (sufficiently larger than a parameter that depends on α_0, w_0). For more details, see [5].

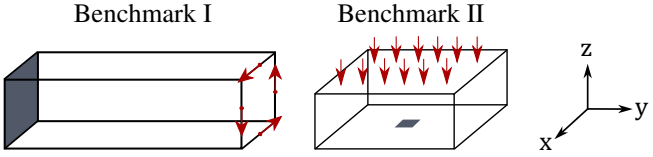
3 The Optimization Algorithm

The level-set method provides a general framework for evolving interfaces based on a implicit representation of domain. The main idea is to represent the optimization domain as a level set of an auxiliary function $\psi : \mathcal{D} \to \mathbb{R}$. The domain Ω and its complementary $\mathcal{D}\backslash\Omega$ are then obtained as $\Omega = \{x \in \mathcal{D} : \psi(x) < 0\}$ and $\mathcal{D}\backslash\Omega = \{x \in \mathcal{D} : \psi(x) > 0\}$. The approach we adopt combines two ingredients. The first is the unconstrained optimization algorithm proposed by Amstutz and Andrä [6], known as the SLERP algorithm, which estabilishes a topological-derivative based strategy to evolve the level-set function. The second is the Null-Space algorithm for constrained optimization, proposed by Feppon [8]. As usual in TO formulations, a regularization filter F_ε is also applied: the characteristic function is replaced with a density function $\rho_\varepsilon := F_\varepsilon(\chi_\Omega)$, which takes intermediate values in a thin region (depending on the parameter ε) around the interface between the two phases. The material properties in the grey areas are described using the SIMP-ALL interpolation, proposed by Ferrer [7]. For the topological derivative of the compliance, volume and first eigenvalue shape functionals, see [9].

4 Numerical Experiments and Final Remarks

Numerical experiments were performed in `MATLAB`, using the `SWAN` package[1]. Two different benchmark problems were considered and the boundary conditions and specific parameters are exhibited in Table 3. The prescribed volume fraction is $V^* = 30\%$. For the eigenvalue problem, we have chosen $w_0 = \alpha_0 = 10^{-3}$ and we have imposed $\lambda_1 > \lambda_1^{\min}$, with $\lambda_1^{\min} = 2.0$. The final designs are illustrated in Table 4. Note that when the connectivity constraints are not imposed, the first eigenvalue associated with the optimal design is close to zero, indicating the presence of void islands, while when the eigenvalue constraints are imposed, the enclosed holes are removed. Note that this comes at the expense of a slight increase in the compliance.

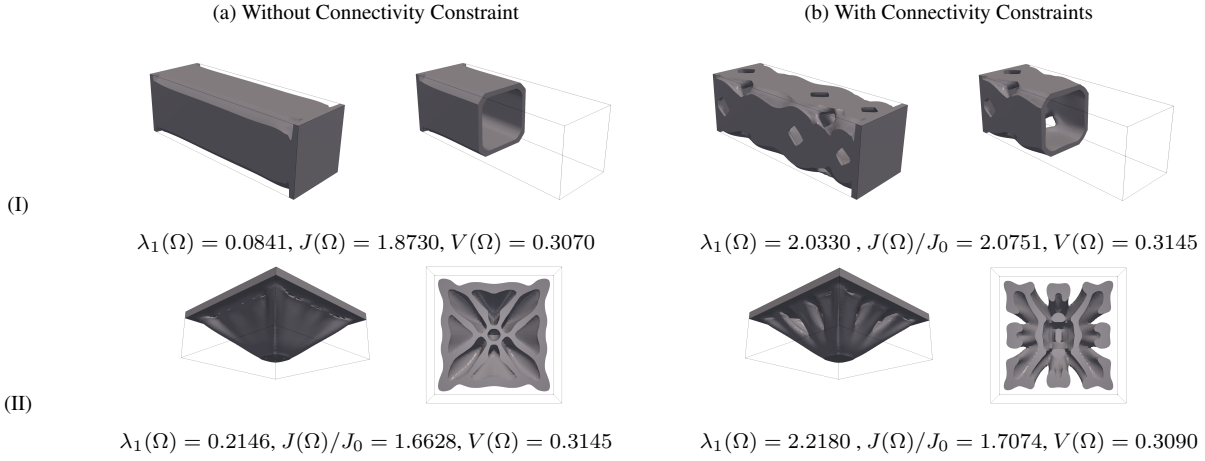
Figure 3. Three dimensional benchmark problems. The grey regions indicate homogeneous Dirichlet boundary conditions, whereas the applied forces are illustrated in red, corresponding to Neumann boundary conditions. For Benchmark I, the domain is $3 \times 1 \times 1$, and the mesh is $90 \times 30 \times 30$. For Benchmark II, one fourth of the domain is optimized; the domain is $1 \times 1 \times 0.75$, and the mesh is $30 \times 30 \times 24$.



As a final remark, let us observe that it is known that when working with eigenvalue formulations with density based methods, phenomenons such as the appearance of *artificial modes* in low density regions can be a challenge

[1] `https://github.com/SwanLab/Swan`

Figure 4. Numerical results obtained. In order to visualize the interior of the designs, a cut in the y-axis is exhibited for Benchmark (I) and in the z-axis for Benchmark (II).



[10]. The addition of heuristics is frequently needed to avoid optimal designs with large grey areas. This issue has not been observed in our level-set based formulation, indicating that it can be particularly well-suited to handle the eigenvalue-based formulation for connectivity constraints. Finally, note that the strategy discussed in this work is independent of the physical model governing the topology optimization formulation, so that it could be directly applied in other contexts. Moreover, although this manuscript is written under the perspective of avoiding void islands, the approach may be easily adapted to handle other type of connectivity constraints.

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