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Two-stage topology optimization of concrete slabs reinforced with steel using a Bidirectional Evolutionary Structural Optimization approach

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Abstract. This paper investigates the application of a two-stage Bidirectional Evolutionary Structural Optimization (BESO) method for the topology optimization of Reinforced Concrete (RC) slabs. The primary objective in a first stage is to achieve a minimum compliance design using only concrete. The second stage then strategically introduces steel reinforcement into highly stressed areas of the optimized concrete structure to improve its structural performance. A fast and efficient finite element analysis (FEA) code implemented in MATLAB, utilizing eight-node hexahedral elements with reduced integration, is employed for the structural analysis and to guide the Soft-Kill BESO process. The self-weight of the structure is considered in the analysis, and the optimization is performed under an approach that can accommodate multiple loading scenarios. A power law penalization scheme is adopted to enforce a clear distinction between solid material-void regions, in the first stage, and between solid-solid during the second stage optimization process. To calibrate the proposed algorithm, a benchmark example was selected from the literature, and the results are compared, demonstrating the accuracy of the implemented approach. The study showcases the capability of this methodology to generate innovative and efficient slab designs with an optimized distribution of concrete and steel, potentially leading to reduced material consumption and enhanced structural efficiency. Visualizations of the optimized material distribution, displacements, and stresses are presented to illustrate the effectiveness of the proposed approach.

Keywords: RC slabs, structural optimization, three-dimensional finite element analysis, BESO, SIMP Penalization Scheme

1. Introduction

Topology Optimization (TO) identifies the most efficient structural layout within a defined domain by strategically adding or removing material to meet load and boundary conditions while minimizing material use. Though a relatively recent approach, it is increasingly applied to complex engineering problems. The Bidirectional Evolutionary Structural Optimization (BESO) method, introduced by Huang and Xie [1], iteratively refines material distribution to enhance structural efficiency. In essence, TO seeks to determine the most efficient material distribution within a design space to achieve specific performance goals, such as maximizing stiffness while minimizing material usage. As highlighted by Jaouadi and Lahmer [2], this process enhances structural efficiency by adhering to predefined constraints. The resulting designs often feature intricate, lightweight geometries that optimize material strength and reduce deformation energy, leading to more effective use of resources.

Several TO methods have been developed and adapted for applications in concrete structures, including Evolutionary Structural Optimization (ESO), as applied by Wang et al. [3]; BESO, as demonstrated by Chen et al. [4] and Amaral et al. [5,6]; Solid Isotropic Material with Penalization (SIMP), as used by Montaute et al. [7]; and level-set methods, as employed by Simonetti et al. [8]. These studies collectively demonstrate the established capability of TO in optimizing concrete structures.

Early TO applications in this field often focused on reinforcement optimization or simplified 2D analyses. For instance, Chaves [9] used density-based methods in Ansys to redistribute carbon-fiber reinforcement in RC slabs, reducing material use by 48% while maintaining stiffness. The study found that optimized layouts aligned with tensile stress zones, though results were highly dependent on boundary conditions. Advancements in computational methods soon expanded TO's scope, enabling more sophisticated modeling. Meanwhile, Young et al. [10] pioneered multi-load case handling via a weighted compliance approach in BESO, though their stress constraints lacked detailed documentation.

Recent studies have leveraged 3D modeling and automation to address complex challenges in Reinforced Concrete (RC) structures. Park et al. [11] applied Topology Optimization (TO) to RC slabs, aiming to mitigate durability issues like water infiltration and rebar corrosion by installing water-blocking blocks at slab edges. Their work optimized material distribution for improved strength-to-weight ratios, combining experimental tests with FEA using the Concrete Damaged Plasticity model in Abaqus. The resulting designs reduced material in low-stress regions while maintaining initial load capacity; however, crack patterns highlighted the need for robust material modeling at the block-slab interface. Similarly, Zhou and Li [12] optimized 2D fiber-reinforced composites by aligning fibers with principal stress directions to minimize shear and promote axial-load-dominated behavior. Further refinements included multi-load truss optimization by Stragiotti et al. [13], which combined linear programming and heuristic methods to minimize volume under stress and buckling constraints. While not directly focused on RC, this work underscores the growing complexity nature of modern TO applications.

Despite these advancements, the comprehensive integration of multi-material optimization with detailed three-dimensional analysis for RC slabs remains an area ripe for further exploration, especially concerning the explicit handling of distinct failure criteria for concrete and steel.

Therefore, this work aims to develop and validate a three-dimensional Topology Optimization model specifically tailored for RC slab applications, with a particular focus on highlighting and implementing a robust multi-material approach. The model integrates the BESO method with Finite Element Analysis (FEA), implemented in MATLAB using 8-node isoparametric hexahedral elements (HEX-8) with 3 degrees of freedom per node and 2×2×2 Gaussian integration. This element type offers a good balance between computational efficiency and accuracy in capturing the structural behavior of solid components, essential for 3D analyses of slabs. Also, the use of this kind of integration helps prevent numerical issues such as volumetric locking, which is important when modeling nearly incompressible materials like concrete. The novelty of this approach lies in the use of a two-stage optimization process: the first stage minimizes structural compliance under volume constraints, and the second replaces regions of overstressed concrete, identified via the Ottosen failure criterion, with steel elements, leveraging the von Mises yield surface. This ensures not only structural efficiency but also safety against material failure by optimally combining the benefits of both concrete and steel in a single structural system. The proposed methodology is demonstrated through two case studies: the first is a three-dimensional deep beam used to validate the algorithm's multi-material and 3D capabilities, and the second involves the optimization of an RC slab to verify the mechanical behavior of the optimized bi-material topology under realistic loading conditions.

2. Theoretical Foundations

The BESO method developed by Huang and Xie [1] iteratively evolves a structure by removing inefficient material and adding material where needed based on element sensitivity. The sensitivity of each element, often defined as the strain energy density (normalized by the total volume constraint), guides this process. Elements with low sensitivity values (which correspond to regions of high local compliance and low contribution to stiffness) are candidates for removal. Conversely, elements with high sensitivity are retained or reinserted, as they significantly contribute to the structural performance. A volume control parameter governs this process, gradually approaching an optimal layout by updating the material distribution based on the relative ability of each element to carry structural loads. In the BESO method for compliance minimization under a volume constraint, the optimization problem is represented in eq. (1).

$$\begin{gathered} Find\ \boldsymbol{x}_j = \left(x_{1j}, x_{2j}, \dots, x_{ij}, \dots, x_{Nj}\right)^T \\ to\ Minimize\ \ C = \boldsymbol{u}^T \boldsymbol{K} \boldsymbol{u} \\ Subject\ to: \begin{cases} V_{j,target} - \sum_{i=1}^{N} x_{ij} V_{f,i} = 0 \\ \boldsymbol{K}\boldsymbol{u} = \mathbf{f}^{\boldsymbol{ext}} \\ Error(C) < \Theta \end{cases} \end{gathered} \tag{1}$$

where $\boldsymbol{x}$ is the design variable vector with N terms, x_{ij} is the material density of the i^{th} element of the j^{th} material used in the material interpolation scheme ($x_{ij} = 1$ for $E \geq E_j$ and $x_{ij} = x_{min}$ for $E \leq E_{j+1}$, where E is the elasticity material modulus, where x_{min} is threshold value to identify the less stiff material within the loop, $V_{j,target}$ is the prescribed volume fraction to reach during the optimization process, $V_{f,i}$ denotes the volume of the i^{th} element, C denotes the compliance (at least for 1/2 , the same as elastic strain energy), Θ is an allowable convergence tolerance, $\boldsymbol{u}$ is the global displacement vector, $\boldsymbol{K}$ is the global stiffness matrix, $\mathbf{f}^{\boldsymbol{ext}}$ is the external force vectors, N is the number of elements used to discretize the design domain. To reduce computational cost in this study, the conjugate gradient method implemented in MATLAB, combined with a modified incomplete Cholesky preconditioner, is employed. Moreover, eq. (1) corresponds to the first stage of the optimization process.

Since the second stage of the algorithm determines the internal steel/concrete distribution iteratively, starting from the optimal topology of concrete and voids from the first stage, a TO formulation with stress constraints is adopted to guide the material replacement process. The stress limits for the concrete and steel materials are approximations, since they are determined by the maximum stress function defined as $\max(\sigma_i) - \sigma_{lim} \leq 0$ (non-differentiable function). The evaluation of the stress weights in the elements is considered based on an empirical exponent q (q-norm, see Le et al. [14]) to make the constraint as smooth as possible $[\sum_{i=1}^{N}(\sigma_i)^q]^{\frac{1}{q}} = \sigma^q \leq \sigma_{lim}$, where σ_i represents the von Mises or Ottosen equivalent stress at the element center. Here the algorithm differs both materials (concrete and steel) and keep exchanging the concrete with high compliance by steel as long as there is any failure with this material. In other words, the algorithm uses two different failure models for concrete (Ottosen's four-parameter criterion) and steel (von Mises yield criterion), iteratively replacing concrete elements with high compliance by steel while detecting any material failure in the finite element mesh [5,6].

To mitigate checkerboard pattern, a filtering technique is introduced that incorporates information from neighboring elements' sensitivities. Initially, a nodal sensitivity value is computed by taking a weighted average of surrounding elements' sensitivity values, where the weights are determined by the distance between each element and the node. This process introduces a minimum length scale in the mesh (*rmin*), effectively highlighting nodes that exert notable influence on local sensitivity. The filter then redistributes these nodal values back into the elemental domain by smoothing them into updated element sensitivity values. Each element's new sensitivity is derived from a weighted average of nearby elements, giving greater emphasis to closer ones: $\alpha_i^e = 1/2\ x_i^{p-1} u_i^T K_i u_i$, where *i* is the iteration index and *e* means the sensitivity at an element. This approach promotes smoother transitions in the design and helps achieve a more physically meaningful and manufacturable topology.

3.1 Material Modelling

The linear theory of elasticity combined with failure criteria is one of the most used material laws in RC analysis. Concrete's mechanical response is complex, making it unlikely that any phenomenological approach could explain all possible variations in the material's behaviour. The four-parameter failure surface proposed by Ottosen [15] for instance, is an adequate criterion for explaining the behaviour of short-time loading in concrete. This criterion involves the stress invariants I_1 and J_2, and the loading angle θ (Oliveira et al. [16]). Thus, Ottosen's criterion can be expressed as follows:

$$f(I_1, J_2, \theta) = c_1 J_2 + \lambda J_2^{0.5} + c_2 I_1 - 1 = 0 \tag{2}$$

where c_1 and c_2 are constants, and λ is a function of $\cos 3\theta$:

$$\lambda = \begin{cases} k_1 \cos\left[\frac{1}{3}\cos^{-1}(k_2 \cos 3\theta)\right] & \text{for } \cos 3\theta \geq 0 \\ k_1 \cos\left[\frac{\pi}{3} - \frac{1}{3}\cos^{-1}(-k_2 \cos 3\theta)\right] & \text{for } \cos 3\theta \leq 0 \end{cases} \tag{3}$$

where $\cos 3\theta = 3\sqrt{3} J_3 / (2\sqrt{J_2^3})$. Therefore, the following constants can be defined as: (*i*) $\alpha = 1/(9k^{1/4})$, (*ii*) $\beta = 1/(3.7k^{1.1})$, (*iii*) $c_1 = 1/(0.7k^{0.9})$ and (*iv*) $c_2 = 1 - 6.8(k - 0.07)^2$ and $k = f_{tm}/f_{cm}$. According to Xie and Steven [18], the model encompasses several earlier models as special cases, e.g., the von Mises model for $c_1 = c_2 = 0$ and $\lambda =$ constant, and the Drucker-Prager model for $c_1 = 0$ and $\lambda =$ constant. According to Chen [17], the model encompasses several earlier models as special cases, e.g, the von Mises model for $c_1 = c_2 = 0$ and $\lambda =$ constant and the Drucker-Prager model for $c_1 = 0$ and $\lambda =$ constant.

For the modelling of reinforcing bars made of steel (isotropic material), the von Mises stress criterion is adequate to represent the yielding of this material. The model considers that the yielding of a ductile material begins when the second invariant of deviatoric stress J_2 reaches a critical value (f_y) (Xie and Steven [18]). Thus, for multi-axial 3D problems, the von Mises stress σ_{vm} is defined as

$$f(\sigma, f_y) = \sqrt{\frac{1}{2}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]} - f_y = 0 \tag{4}$$

The problems presented in the following sections are all-static and use the BESO method. They are modelled in three dimensions, and optimize for one load case. The first example was taken from. Borges et al. [19] and the second was a new implementation create by the authors.

3. Numerical Examples

4.1 Verification of the Proposed 3D Algorithm through a Simply Supported Deep Beam

This benchmark problem, based on the deep beam proposed by Goodchild [20], was selected to verify the implementation of the 3D finite element-based topology optimization code. The structure consists of a simply supported deep beam subjected to a distributed load (Figure 1). To represent the materials, the elastic modulus was set to 33 GPa for concrete and 210 GPa for steel, with corresponding Poisson's ratios of 0.22 and 0.30. The compressive and tensile strengths adopted for the concrete were f_c=35 MPa and f_t=3.5 MPa, respectively, while the steel yield strength was f_y=250 MPa. Material densities were assumed as $\rho_{conc} = 2400\ kg/m^3$, $\rho_{steel} = 7800\ kg/m^2$, and a minimum void density of $\rho_{min} = 0.1\ kg/m^3$. Since this problem was previously replicated by Borges et al. [19], who conducted a sensitivity analysis to calibrate the BESO method's parameters for RC optimization, a volume fraction $V_{frac} = 0.6$, with evolutive rates (ER) of 0.05 and 0.01 for the first and second stages, respectively. The penalization factor is set to $p = 3$. A sensitivity filtering scheme is employed to ensure mesh independence and mitigate numerical instabilities. This filter uses a radius defined as $r_{min} = 4 \times L$ during the concrete optimization stage (first stage), and $r_{min} = 2 \times L$ during the steel redistribution stage (second stage), where L represents the average edge length of the finite elements. The mesh is composed of 3D finite elements. The results compared with the literature are presented in Figures 2 and 3 and in Table 1.

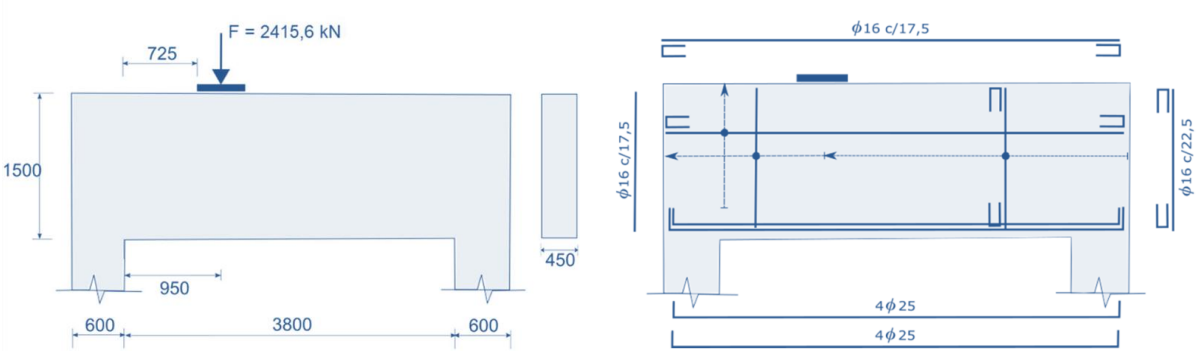


Figure 1. (a) Domain, boundary conditions, loadings and (b) design by Strut-and-tie method.

Table 1. Comparison of the main characteristics of the two problems

Features	This Work (3D Bi-Material)	Borges et al. [19] (2D Bi-Material)
Dimensionality	3D (hexahedral elements)	2D (plane elements)
Element Type	8-node Hexahedral (H8)	4-node plane elements
Mesh (Elements)	100×30×5=15,000	100×30=3,000
Final Compliance	768 Nm	902 Nm
Concrete Vol. Fraction	60%	60%
Failure Criteria	Ottosen (concrete), von Mises (steel)	Ottosen (concrete), von Mises (steel)
Self-weight	Yes	Yes

Table 1 shows that the final compliance obtained in the 3D analysis (768 N.m) is significantly lower than that in the 2D model (902 N.m), indicating a stiffer and better-efficient structural design. The reduction in compliance of approximately 14.86% reflects the advantage of the 3D formulation in capturing out-of-plane effects and optimizing material allocation precisely. Furthermore, the higher mesh resolution of the 3D model and its use of hexahedral elements enable a more precise representation of complex stress states, including torsional and transverse stresses, which cannot be fully captured in 2D plane stress analyses.

As Figures 2(a) and 2(c) show, the resulting topologies reveal that the 3D configuration closely resembles the 2D solution in terms of overall structural layout and primary load paths. In other words, the results of the algorithm suggest that steel reinforcement is required at the bottom of the continuous deep beam spans from both analyses. This behavior can be explained by the presence of tensile stress regions between the beam supports. As Amaral et al. [5] observed, principal stress analysis shows that compressive zones develop near load application and supports because a larger amount of concrete is allocated there due to its high compressive strength. Conversely, tensile stresses dominate in the spans between supports. Because concrete has low tensile strength, steel reinforcement is necessary in these regions to withstand tensile forces and ensure the beam's structural safety. Concerning the stress analysis (Figures 2(b) and 2(d)), both studies demonstrate similar stress distribution patterns,

effectively identifying the regions with stress concentration. The equivalent stress fields in both cases consistently highlight these critical zones, confirming the reliability of the results. Furthermore, the stress levels observed throughout the structures remain below the allowable design limits for the materials, ensuring the structural safety and integrity under the applied loading conditions.

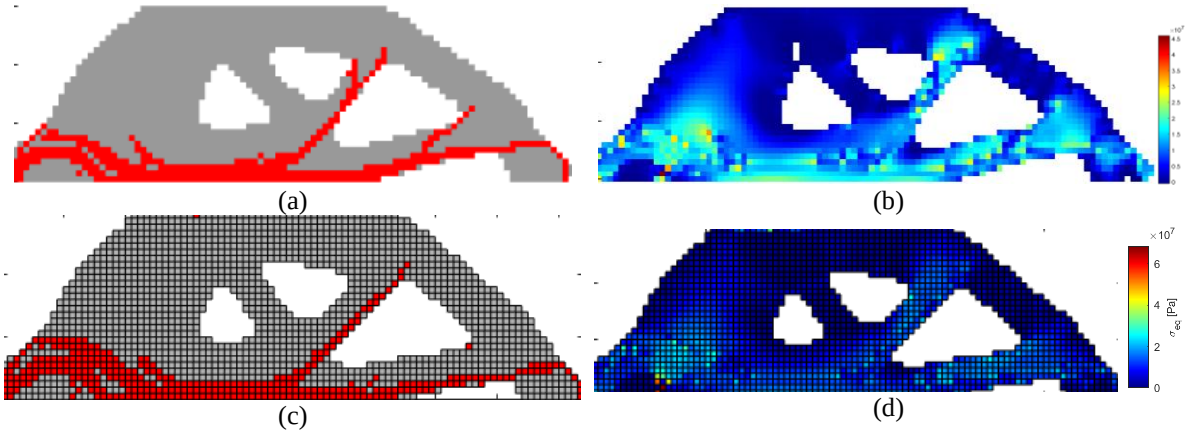


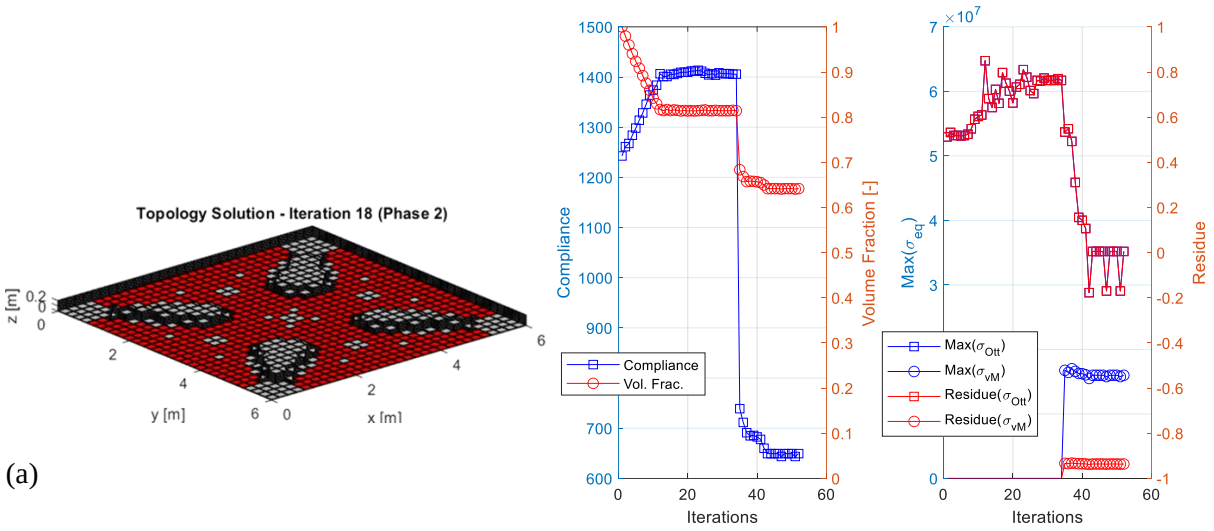
Figure 2. Comparison of final topologies and equivalent stress fields: (a–b) 2D results from Borges et al. [19], and (c–d) 3D results from the present study.

4.2 3D Slab with Distributed Load and Stress Constraints

Problem 2, as depicted in Figure 3(a), considers a RC slab with dimensions 6.0×6.0×0.2 meters (length×width×thickness). Eight-node isoparametric hexahedral 3D Finite Elements are employed, utilizing a mesh size of 30 × 30 × 10 (X, Y, Z respectively) elements, resulting in a total of 9,000 elements. The reason for the relatively large XY element size (20 cm) is to save computational time, which was 120.4722 seconds for this experiment. The slab is supported at its four bottom corners, where all translational degrees of freedom (x, y, z) are fixed. A uniform distributed load of 10,000 N/m² is applied downwards on the entire top surface of the slab. Additionally, the self-weight of the structure is considered, with a gravitational acceleration of 9.8 m/s² factored by 1.35.

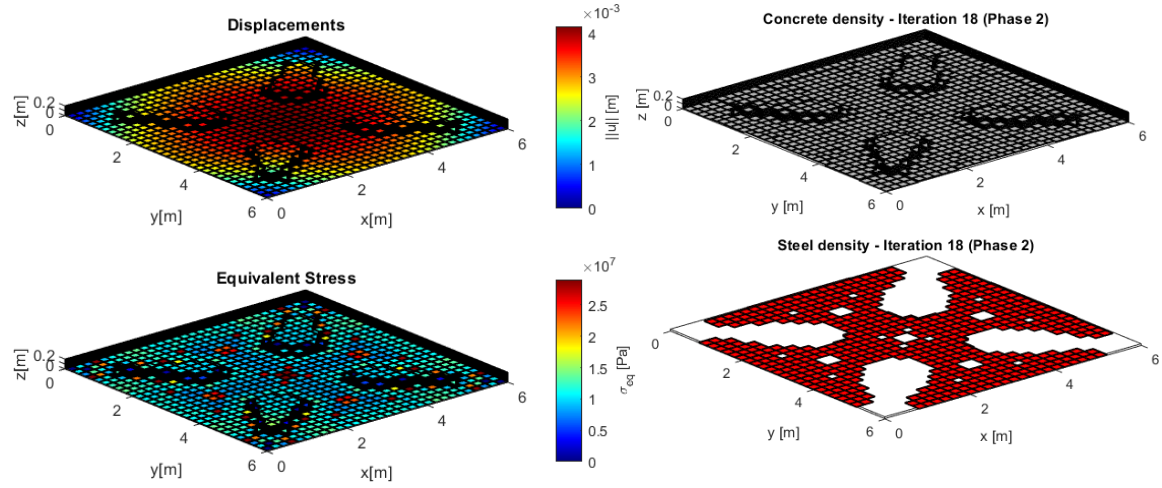
The optimization is performed for a bi-material system, consisting of two stages: a concrete stage and a steel stage. The material properties are interpolated based on the density variable x. For the concrete stage (Stage 1), the Young's modulus (E) varies from 1 Pa to 40 GPa, Poisson's ratio (ν) from 0.01 to 0.22, and density (ρ) from 0 to 2400 kg/m³. For the steel stage (Stage 2), E varies from 40 GPa to 210 GPa, ν from 0.22 to 0.30, and ρ from 2400 to 7800 kg/m³. The concrete and steel have the same strengths as the previous example. The objective function for this problem remains the Compliance. The optimization aims to meet specific stress constraints for both materials: For the steel stage, the von Mises stress (σ_{vM}) must not exceed a yield stress limit of 250 MPa. For the concrete stage, the stress state is checked against the Ottosen criterion, with a uniaxial compressive strength f_{cm}=35 MPa and an uniaxial tensile strength f_{tm}= 3.5 MPa. Parameters for the Ottosen criterion (k, α, β, c_1, c_2) are derived from these strengths as per the fib (International Federation for Structural Concrete) model. The optimization parameters utilized include: a penalization power p= 3 for density interpolation, a stress proportion exponent q=4, and an inertia weight α=0.5 for smoothing sensitivities. The filter radius $rmin_{stage}$ is set to 3 times the average element edge size for the concrete stage and 2 times for the steel stage. The minimum density allowed for elements is $x_{min} = 1\times10^{-3}$. The maximum number of iterations is $iter_{max} = 100$, with a convergence tolerance tol=1×10^{-4}. The volumetric fractions targeted at 80% for the concrete (stage 1), the criterion involves adding material until the equivalent stresses remain below the material's limit, reflecting the design intent for a concrete structure with integrated steel reinforcement. The evolutionary volume ratio er =0.02 for both stages.

Figure 3(a) shows the final bi-material topology for the 3D slab, showing concrete in gray and steel in red, at the end of stage 2. Compliance, volume fraction, maximum stress (concrete and steel) and Stress residuals convergence for the 3D slab results are presented in Figure 3(b). The final Compliance obtained was 649.45 Nm. Figure 4(a) shows the final displacements and equivalent stresses. It is possible to observe that the maximum stress (in all elements) did not exceed the established yield strength limit f_{yd}. Figure 4(b) illustrates the final topology, distinguishing between concrete elements (gray) and steel elements (red). It is noticed that the steel is located in areas subjected to higher tensile stresses, thereby reinforcing the structure at its most critical points.



(a) (b)

Figure 3. (a) Bottom view of final topology (concrete and steel), at the end of stage 2. (b) Compliance, Volume fraction, maximum stress (concrete and steel) and Stress residuals convergence of Problem 2



(a) (b)

Figure 4. (a) Bottom views of displacements of the 3D slab and maximum equivalent stress in the structure in Problem 2. (b) Bottom views for the final topology, separated into concrete (gray) and steel (red) parts

4. Conclusions

This work successfully applied a two-stage bi-material Topology Optimization (TO) methodology to 3D Reinforced Concrete (RC) structures. The method incorporated distinct failure criteria for concrete (Ottosen) and steel (von Mises), and was implemented in MATLAB combined with a (BESO) approach, proving efficient in obtaining robust solutions. The analysis of the Goodchild [20] beam problem in 3D offered a more comprehensive perspective on the optimal material distribution and reinforcement compared to existing 2D solutions in the literature [19].

The explicit comparison with Borges et al. [19] confirmed that while general concrete topologies may be similar, 3D modeling is crucial for a more accurate evaluation of the stress state and a more realistic three-dimensional distribution of steel reinforcement. This ability to consider all stress tensor components and volumetric effects contributes to a safer and more efficient RC design. The successful application to 3D slab optimization, an intrinsically three-dimensional problem, reinforced the methodology's versatility and potential for complex structural problems. Although the work focused on single loading scenarios, the algorithm inherently supports multiple loading conditions, enhancing its applicability for diverse real-world designs. Despite the inherent increase in computational cost of 3D modeling, the resulting wealth of detailed information and enhanced

realism amply justify its application to improve complex civil engineering structures, with the generated topologies serving as a starting point for real-world designs and advanced manufacturing.

Future research will expand the proposed algorithm to: (*i*) consider material non-linearity, (*ii*) multiple load cases during the optimization process, (*iii*) a mesh refinement study to obtain accurate results, (*iv*) explore alternative objective functions (e.g., stress), (*v*) considers other design constraints, limiting natural frequencies, compliance, displacements, fracture energy (to reduce cracking formation) and (*vi*) validation via experiments or commercial software.

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