XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING

A Multiscale Hybrid-Mixed Method with Local Stabilization

Larissa Martins[1], Juan F. Pacazuca[1], Frédéric Valentin[1],

[1]*National Laboratory for Scientific Computing*
Av. Getulio Vargas, 333, Petrópolis, RJ, Brazil
larissam@posgrad.lncc.br, juanpaca@posgrad.lncc.br, valentin@lncc.br

Abstract. We introduce the MHM-UNUSUAL (MHM-UN) method, a stabilized variant of the Multiscale Hybrid-Mixed (MHM) framework, designed to enhance the accuracy and efficiency of local multiscale computations in reaction-dominated and highly heterogeneous problems like porous media. The method incorporates the Unusual Stabilized Finite Element Method (UNUSUAL) into the local problem solvers of MHM, leveraging a residual-based stabilization mechanism that mitigates spurious oscillations without adding excessive numerical diffusion, allowing the use of coarser local meshes while maintaining accuracy. This work extends the theory of the UNUSUAL method to address the local problems arising in the MHM-UN formulation. Numerical results on challenging problems, including boundary layer and the SPE10 benchmark, shows significant improvements over standard Galerkin-based local solvers approaches in precision.

Keywords: Finite element method, Heterogeneus media, Stabilized method, Multiscale Hybrid-Mixed method, Reaction-difussion model.

1 Introduction

The solution of many models can exhibit multiscale behavior, meaning there may be significant variation over small spatial regions or short time intervals. This is common in a wide range of problems where the physical properties of the medium are highly heterogeneous or when the solution presents boundary layers. Standard numerical methods often struggle to approximate such solutions effectively while maintaining a balance between accuracy and computational cost. In this context, numerical methods specifically designed to handle multiscale behavior have emerged as a promising alternative.

Multiscale finite element methods have emerged as powerful alternatives, capable of delivering accurate approximations on coarse meshes without requiring additional assumptions about the solution. Since the seminal work of Babuška and Osborn [1], a vast literature has been developed around this topic. In the context of the Laplace problem, numerous approaches have been proposed over the past two decades, including MsFEM of Hou et al. [2], Efendiev et al. [3], the VMS method of Hughes [4], and the Multiscale Mortar Mixed FEM of Arbogast et al. [5], among others.

Here we will focus on the Multiscale Hybrid-Mixed (MHM) method presented by Harder et al. [6], Araya et al. [7]. The MHM method arises from a hybrid formulation defined at the continuous level on a coarse partition of the domain. It is based on decomposing the exact solution into global and local components. After discretization, this decomposition naturally results in a decoupling of the global and local problems: the global problem involves only the degrees of freedom on the skeleton of the coarse mesh, while the local problems are used to construct the multiscale basis functions. A key feature of the method is that these basis functions can be computed independently and locally, using, for example, standard Galerkin or mixed methods, making the overall approach highly parallelizable and computationally efficient.

The use of the Galerkin approach in the local problems is often chosen because it is classical and computationally attractive. However it may produce spurious oscillations when the equations are singularly perturbed as seen in Franca and Valentin [8] if the local meshes are not fine enough, which increases computational costs. Another alternative would be to use stabilized local solvers, in particular, we propose to use the the Unusual Stabilized Finite Element Method (UNUSUAL) of Franca and Valentin [8].

In this work, we propose a numerical scheme named MHM-UNUSUAL (MHM-UN), which offers: (i) accurate solutions on coarse local and global meshes, (ii) optimal convergence in standard norms, and (iii) robustness in highly heterogeneous media. To this end, we extend the UNUSUAL method to handle variable coefficients and Neumann boundary conditions, since the original UNUSUAL formulation is limited in scope, as it does not naturally offers this type of support, in order to use this method as local solver.

The remainder of this paper is organized as follows: Section 2 introduces the model problem, notation, and the characterization of the solution based on local and global problems. Section 3 describes the discretization framework, including mesh construction, functional spaces, key assumptions, and the extension of the UNUSUAL method for the local solvers, culminating in the MHM-UN formulation. Section 4 presents numerical experiments, and Section 5 offers concluding remarks.

2 Settings and Preliminary results

2.1 Model problem

Let $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, be an open, bounded, polytopal domain with Lipschitz boundary $\partial\Omega$. Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, we consider the following boundary value problem: find $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = g$ such that

$$\int_\Omega \sigma uv \, d\boldsymbol{x} + \int_\Omega \mathcal{A}\nabla u \cdot \nabla v \, d\boldsymbol{x} = \int_\Omega fv \, d\boldsymbol{x}, \quad \text{for all } v \in H_0^1(\Omega). \tag{1}$$

Here, $\sigma \in L^\infty(\Omega)$ is a reaction coefficient and $\mathcal{A} \in L^\infty(\Omega)^{d\times d}$ is a symmetric diffusion matrix and that may involve multiscale features and it is supposed to be bounded and uniformly elliptic in Ω.

2.2 Hybridization

Following the idea of Barrenechea et al. [9], we start introducing $\mathscr{P}$, a collection of closed, bounded, disjoint polytopes, K, such that $\bar{\Omega} = \cup_{K\in\mathscr{P}} K$. The shapes of the polytopes K are, *a priori*, arbitrary, but we suppose that they satisfy a minimal angle condition. The diameter of K is $\mathcal{H}_K$ and we denote $\mathcal{H} = \max_{K\in\mathscr{P}} \mathcal{H}_K$. For each $K \in \mathscr{P}$, $\boldsymbol{n}^K$ denotes the unit outward normal to ∂K, such that $\boldsymbol{n}^K = \boldsymbol{n}$ on $\partial\Omega$ where, $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. We also introduce $\partial\mathscr{P}$ as the set of boundaries ∂K, $\mathcal{E}$ the set of the faces in $\mathscr{P}$, and $\mathcal{E}_0$ the set of internal faces. By $\boldsymbol{n}_E$, we denote a unit normal vector on faces $E \in \mathcal{E}$, and $\boldsymbol{n}_E^K$ the unit outward normal vector on E with respect to K.

Now, given a partition $\mathscr{P}$ of Ω we define the spaces

$$V := H^1(\mathscr{P}) := \{v \in L^2(\Omega) : v|_K \in H^1(K), \, \forall K \in \mathscr{P}\}, \quad \Lambda := \left\{\boldsymbol{\tau} \cdot \boldsymbol{n}^K|_{\partial K} : \boldsymbol{\tau} \in H(\text{div}, \Omega), \, \forall K \in \mathscr{P}\right\}. \tag{2}$$

Over the spaces V and Λ we define the respective norms

$$\|v\|_V := \left\{\sum_{K\in\mathscr{P}} \frac{1}{d_\Omega^2}\|v\|_{0,K}^2 + \|\nabla v\|_{0,K}^2\right\}^{1/2} \text{ and } \|\mu\|_\Lambda := \inf_{\substack{\boldsymbol{\tau}\in H(\text{div},\Omega) \\ \boldsymbol{\tau}\cdot\boldsymbol{n}^K=\mu \text{ on } \partial K}} \|\boldsymbol{\tau}\|_{\text{div},\Omega}\,, \tag{3}$$

where $d_\Omega > 0$ is the diameter of Ω and $H(\text{div}, \Omega) := \{\boldsymbol{\tau} \in L^2(\Omega)^d \, : \, \nabla \cdot \boldsymbol{\tau} \in L^2(\Omega)\}$ with norm

$$\|\boldsymbol{\tau}\|_{\text{div}} := \left(\sum_{K\in\mathscr{P}} \|\boldsymbol{\tau}\|_{0,K}^2 + d_\Omega^2\|\nabla \cdot \boldsymbol{\tau}\|_{0,K}^2\right)^{1/2}. \tag{4}$$

We define the broken products on $\mathscr{P}$ and $\partial\mathscr{P}$ as

$$(v, w)_\mathscr{P} := \sum_{K\in\mathscr{P}} (v, w)_K \quad \text{and} \quad \langle\mu, v\rangle_{\partial\mathscr{P}} := \sum_{K\in\mathscr{P}} \langle\mu, v\rangle_{\partial K}\,, \tag{5}$$

where $(\cdot, \cdot)_D$ denotes the $L^2(D)$ inner product, and $\langle\cdot, \cdot\rangle_{\partial D}$ denotes the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D)$, for an open and measurable subset $D \subset \mathbb{R}^d$. We are ready to present a hybrid formulation for eq. (1). Here we relax the continuity of u on the skeleton $\partial\mathscr{P}$ by introducing the Lagrange multiplier λ. The hybrid formulation reads: Find $(u,\lambda) \in V \times \Lambda$ such that

$$\begin{cases} (\sigma u, v)_{\mathscr{P}} + (\mathcal{A}\nabla u, \nabla v)_{\mathscr{P}} + \langle \lambda, v\rangle_{\partial\mathscr{P}} = (f,v)_{\mathscr{P}}, & \text{for all } v \in V, \\ \langle \mu, u\rangle_{\partial\mathscr{P}} = \langle \mu, g\rangle_{\partial\Omega}, & \text{for all } \mu \in \Lambda. \end{cases} \tag{6}$$

The formulation eq. (6) is equivalent to eq. (1) as proven in Araya et al. [7].

2.3 A characterization of the exact solution

The exact solution of eq. (1) can be characterized in terms of the solution to local and global problems. To this end, following closely Araya et al. [10], we define the bounded mappings $T \in \mathcal{L}(\Lambda, V)$ and $\hat{T} \in \mathcal{L}(L^2(\Omega), V)$ as follows:

- for all $\mu \in \Lambda$, $T\mu|_K \in H^1(K)$ is the unique solution of

$$(\sigma T\mu, v)_K + (\mathcal{A}\nabla T\mu, \nabla v)_K = -\langle \mu, v\rangle_{\partial K} \quad \text{for all } v \in H^1(K) \text{ and } K \in \mathscr{P}\,; \tag{7}$$

- for all $q \in L^2(\Omega)$, $\hat{T}q|_K \in H^1(K)$ is the unique solution of

$$(\sigma \hat{T}q, v)_K + (\mathcal{A}\nabla \hat{T}q, \nabla v)_K = (q, v)_K \quad \text{for all } v \in H^1(K) \text{ and } K \in \mathscr{P}\,. \tag{8}$$

Hence, the solution of eq. (1) can be written as $u = T\lambda + \hat{T}f$ where $\lambda \in \Lambda$ solves the following problem

$$-\langle \mu, T\lambda\rangle_{\partial\mathscr{P}} = \langle \mu, \hat{T}f\rangle_{\partial\mathscr{P}} - \langle \mu, g\rangle_{\partial\Omega} \quad \text{for all } \mu \in \Lambda. \tag{9}$$

The well-posedness of eq. (9) was proved in Araya et al. [7].

3 The MHM-UNUSUAL Method

3.1 Partitions and spaces

The MHM method uses a multi-level discretization starting from the first-level partition $\mathscr{P}$. Notably, each face $E \in \mathcal{E}$ and polytopal element $K \in \mathscr{P}$ may carry its own family of partitions in a way that each member is *a priori* independent of each other (cf. Gomes et al. [11]). We start discretizing the set of faces $E \in \mathcal{E}$. For this, let $\{\mathcal{E}_H\}_{H>0}$ be a family of partitions of $\mathcal{E}$, for which each $E \in \mathcal{E}$ is split into faces F of diameter $H_F \leq H := \max_{F\in\mathcal{E}_H} H_F$. We call $\mathcal{E}_H^K$ the collection of faces $F \in \mathcal{E}_H$ such that $F \subset \partial K$. For the second-level discretization we assume the following:
Assumption A: For each $K \in \mathscr{P}$, there exists $\left\{\Xi_H^K\right\}_{H>0}$ a shape-regular family of conforming simplicial partitions of K matching with $\{\mathcal{E}_H\}_{H>0}$, i.e., for each $F \in \mathcal{E}_H^K$ there exists an element $\kappa \in \Xi_H^K$ such that $\partial\kappa \cap \partial K = F$.

The triangulation Ξ_H is used for theoretical construction. To enable practical implementation, we define a *red refinement* as the division of each triangle in Ξ_H^K into four sub-triangles by connecting edge midpoints. Let $\{\mathscr{T}_h^K\}_{h>0}$ be a family of shape-regular simplicial meshes defined by regular refinements of the red-refined minimal triangulation of Ξ_H^K, based on Assumption A. The global mesh is $\mathscr{T}_h := \bigcup_{K\in\mathscr{P}} \mathscr{T}_h^K$, with global mesh size $h := \max_{\kappa\in\mathscr{T}_h} h_\kappa$.

For $k \geq 1$ and $\ell \geq 0$ we define the following finite element spaces associated to $\mathcal{E}_H$ and $\mathscr{T}_h^K$

$$V_h := \prod_{K\in\mathscr{P}} V_h(K)\,, \quad \text{where} \quad V_h(K) := \{v_h \in C^0(K) \,:\, v_h|_T \in \mathbb{P}_k(T),\, \forall\, T \in \mathscr{T}_h^K\}\,, \tag{10}$$

$$\Lambda_H := \{\mu_H \in \Lambda : \mu_H|_F \in \mathbb{P}_\ell(F),\, \forall\, F \in \mathcal{E}_H\}\,. \tag{11}$$

3.2 Extension of the UNUSUAL Method

In this section we propose to solve the local problems eq. (7) and eq. (8) with an extension of the method from Franca and Valentin [8], that can handle mixed boundary conditions and variable coefficients. For that we consider the following weak formulation: Find $u \in H^1_{0,D}(\Omega)$ such that

$$(\sigma u, v)_\Omega + (\mathcal{A}\nabla u, \nabla v)_\Omega = (f, v)_\Omega + \langle r, v\rangle_{\partial\Omega_N} \quad \forall v \in H^1_{0,D}(\Omega), \tag{12}$$

where $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$, $r \in H^{-1/2}(\partial\Omega_N)$, and $H^1_{0,D}(\Omega) := \{u \in H^1(\Omega) : u|_{\partial\Omega_D} = 0\}$.

Consider a simplicial partition $\mathscr{T}_h$ of Ω. Define the finite element space

$$W_h := \{v \in C^0(\overline{\Omega}) : v|_K \in \mathbb{P}_k(K) \text{ for all } K \in \mathscr{T}_h\} \subset H^1_{0,D}(\Omega), \tag{13}$$

with polynomial degree $k \geq 1$. The **UNUSUAL** method reads: Find $u_h \in W_h$ such that

$$\begin{aligned}(\sigma u_h, v_h)_\Omega + (\mathcal{A}\nabla u_h, \nabla v_h)_\Omega - &\sum_{K\in\mathcal{T}_h} \big(\sigma u_h - \nabla\cdot(\mathcal{A}\nabla u_h), \tau_K(\sigma v_h - \nabla\cdot(\mathcal{A}\nabla v_h))\big)_K \\ &= (f, v_h)_\Omega + \langle r, v_h\rangle_{\partial\Omega_N} - \sum_{K\in\mathcal{T}_h} (f, \tau_K(\sigma v_h - \nabla\cdot(\mathcal{A}\nabla v_h)))_K \quad \forall v_h \in W_h\,,\end{aligned} \tag{14}$$

where the stabilization parameter on each $K \in \mathscr{P}$ is

$$\tau_K = \frac{m_k h_K^2}{\sigma^K_{\text{max}} m_k h_K^2 \max\{Pe_K, 1\} + 2\mathcal{A}^K_{\text{min}}}\,, \tag{15}$$

with $\sigma^K_{\text{max}} = \sup_{\boldsymbol{x}\in K} |\sigma(\boldsymbol{x})| \leq \sigma_{\text{max}}$ and $\mathcal{A}^K_{\text{min}}|\boldsymbol{y}|^2 \leq \mathcal{A}(\boldsymbol{x})\boldsymbol{y}\cdot\boldsymbol{y} \leq \mathcal{A}^K_{\text{max}}|\boldsymbol{y}|^2$, for all $\boldsymbol{x} \in K, \boldsymbol{y} \in \mathbb{R}^d$ and $m_k := 1/3$ for piecewise linears. The local Péclet number is defined as

$$Pe_K = \frac{2\mathcal{A}^K_{\text{min}}}{m_k \sigma^K_{\text{max}} h_K^2}\,. \tag{16}$$

3.3 The MHM-UN Method

We approximate the local problems eq. (7) and eq. (8) using the UNUSUAL method introduced in eq. (14). More specifically, for $K \in \mathscr{P}$:

- for all $\mu \in \Lambda$, the function $T_h\mu \in V_h(K)$ is the unique solution of

$$\begin{aligned}(\sigma T_h\mu, v_h)_K + (\mathcal{A}\nabla T_h\mu, \nabla v_h)_K - &\sum_{\kappa\in\mathscr{T}_h^K} \big(\sigma T_h\mu - \nabla\cdot(\mathcal{A}\nabla T_h\mu),\ \tau_\kappa(\sigma v_h - \nabla\cdot(\mathcal{A}\nabla v_h))\big)_\kappa \\ &= -\langle\mu, v_h\rangle_{\partial K}, \quad \text{for all } v_h \in V_h(K) \text{ and } K \in \mathscr{P}\,;\end{aligned} \tag{17}$$

- for all $f \in L^2(\Omega)$, the function $\hat{T}_h f \in V_h(K)$ is the unique solution of

$$\begin{aligned}(\sigma\hat{T}_h f, v_h)_K + (\mathcal{A}\nabla\hat{T}_h f, \nabla v_h)_K - &\sum_{\kappa\in\mathscr{T}_h^K} \big(\sigma\hat{T}_h f - \nabla\cdot(\mathcal{A}\nabla\hat{T}_h f),\ \tau_\kappa(\sigma v_h - \nabla\cdot(\mathcal{A}\nabla v_h))\big)_\kappa \\ &= (f, v_h)_K - \sum_{\kappa\in\mathscr{T}_h^K} (f,\ \tau_\kappa(\sigma v_h - \nabla\cdot(\mathcal{A}\nabla v_h)))_\kappa, \quad \text{for all } v_h \in V_h(K) \text{ and } K \in \mathscr{P}\,.\end{aligned} \tag{18}$$

Both discrete local problems eq. (17) and eq. (18) are well-posed and define continuous operators. The MHM-UN method reads: Find $\lambda_H \in \Lambda_H$ such that

$$-\langle \mu_H, T_h \lambda_H \rangle_{\partial \mathscr{P}} = \langle \mu_H, \hat{T}_h f \rangle_{\partial \mathscr{P}} - \langle \mu_H, q \rangle_{\partial \Omega} \quad \text{for all } \mu_H \in \Lambda_H. \tag{19}$$

The approximated solution is given by

$$u_{Hh} := T_h \lambda_H + \hat{T}_h f. \tag{20}$$

Remark 1. When $\tau_\kappa = 0$ for all $\kappa \in \mathscr{T}_h^K$ and all $K \in \mathscr{P}$, the method is referred to as the MHM-Galerkin method (MHM-Gal), proposed in some works of Araya et al. [7], Gomes et al. [11], Fernando et al. [12]. □

We now state the main convergence result for the MHM-UN method:

Theorem 1. Suppose that assumption A holds, $k \geq \ell + d$ or $k = \ell + 1$ and $\mathscr{T}_h^K$ has one red refinement for each $K \in \mathscr{P}$. Then the method eq. (19) is well-posed. Furthermore, let u be the solution of eq. (1) such that $u \in H^{k+1}(\mathscr{P})$ and $\mathcal{A}\nabla u \in H^{\ell+1}(\mathscr{P})^d$. Then the approximation u_{Hh} satisfies

$$\|u - u_{Hh}\|_V \leq C \left(H^{\ell+1} |\mathcal{A}\nabla u|_{\ell+1,\mathscr{P}} + h^k |u|_{k+1,\mathscr{P}} \sum_{K \in \mathscr{P}} \sum_{\kappa \in \mathscr{T}_h^K} C_\kappa \left(H(1 - Pe_\kappa) h_\kappa + H(Pe_\kappa - 1) \right) \right), \tag{21}$$

where C is independent of h, H and $\mathcal{H}$, C_κ is independent of h_κ for each $\kappa \in \mathscr{T}_h^K$ and $H(\cdot)$ stands for the Heaviside function. □

4 Numerical Results

This section presents two numerical experiments. The first assesses the theoretical aspects of the MHM-UN method. The second demonstrates its robustness when applied to problems with highly heterogeneous coefficients.

4.1 Analytical Test Case

We consider problem eq. (12) with $f = 1.0$ and an isotropic diffusion tensor $\mathcal{A} = \epsilon I_2$, where $\epsilon, \sigma > 0$ are constants. The computational domain is $\Omega = (0,1)^2$, with homogeneous Dirichlet and Neumann boundary conditions. The analytical solution is

$$u(x,y) := \frac{1}{\sigma} \left[\frac{\sinh\left(\sqrt{\frac{\sigma}{\epsilon}}(x-1)\right)}{\sinh\left(\sqrt{\frac{\sigma}{\epsilon}}\right)} - \frac{\sinh\left(\sqrt{\frac{\sigma}{\epsilon}}x\right)}{\sinh\left(\sqrt{\frac{\sigma}{\epsilon}}\right)} + 1 \right], \tag{22}$$

which exhibits a sharp boundary layer as $\epsilon \to 0$.

Figure 1 show the $\mathcal{H}-$convergence results for in the $L^2(\Omega)-$ and $H^1(\Omega)-$norm for $\epsilon = \sigma = 1.0$, where no boundary layer is present. As we predicted by the theory, the $H^1(\Omega)-$norm converges with an optimal order of $\mathcal{O}(\mathcal{H}^{\ell+1})$.

For a more challenging scenario, we assess the performance of the methods as $\epsilon \to 0$. Figure 2 presents a comparison of the solutions obtained by MHM-UN, MHM-Gal, and UNUSUAL using a coarse mesh with 512 triangles, a partition size $H = \mathcal{H}/4$, and local mesh size $h = H$. The MHM-UN method successfully captures the boundary layer with high accuracy, whereas the MHM-Galerkin method exhibits noticeable oscillations. The UNUSUAL method yields a stable solution but fails to resolve finer features of the boundary layer.

4.2 Heterogeneous Test Case: SPE-10 (Layer 36)

We consider Model 2 from the Society of Petroleum Engineers Comparative Solution Project (cf. Jaramillo et al. [13]), hereafter referred to as the SPE-10 model. To validate our method, we select layer 36 due to its extreme heterogeneity. The computational domain is $\Omega = (0,1200) \times (0,2200)$, with Dirichlet conditions $u(x,0) = 1$ (bottom) and $u(x,2200) = 0$ (top), and homogeneous Neumann conditions elsewhere. The reaction is $\sigma := 1.0$ and the diffusion tensor is defined as $\mathcal{A}(x,y) := \kappa(x,y) I_2$, where $\kappa(x,y)$ corresponds to the permeability field from layer 36, shown in Figure 3 (left).

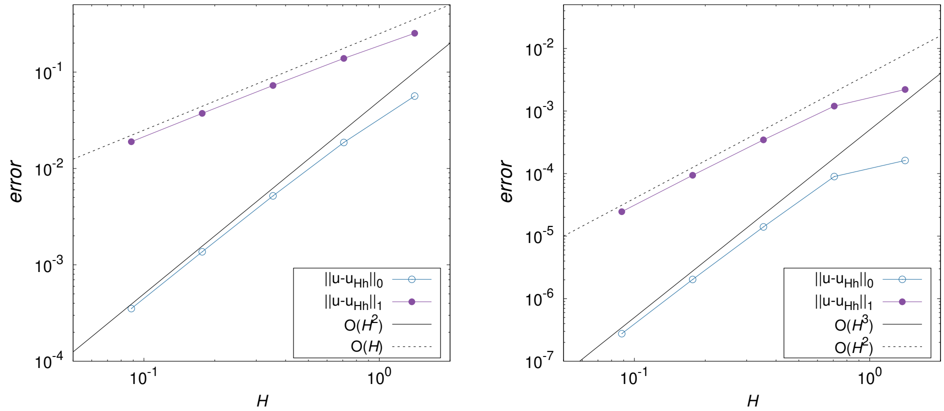

Figure 1. Convergence with $\ell = 0$, $k = 1$, and $h = \mathcal{H}$ (left) and the convergence with $\ell = 1$, $k = 2$, and $h = \frac{\mathcal{H}}{2}$ (right).

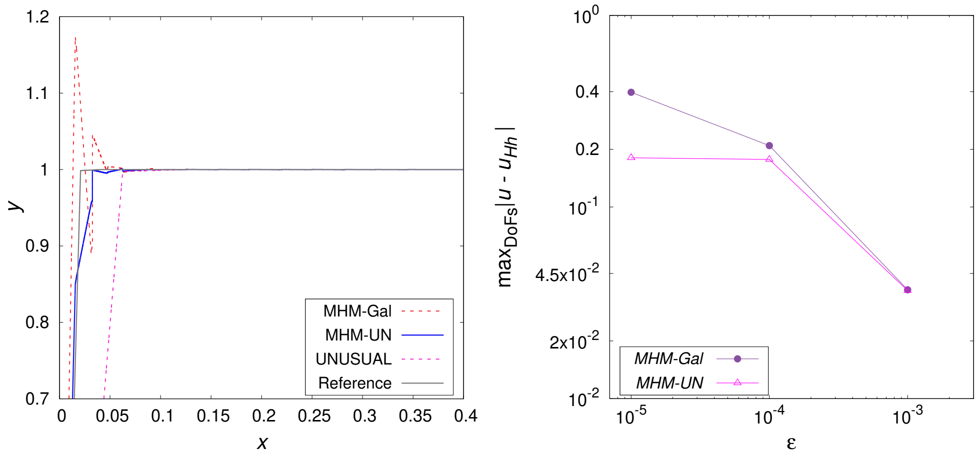

Figure 2. Comparison along $y = 0.47$ for MHM-UN, MHM-Gal ($\ell = 0$, $k = 1$) and UNUSUAL (left) when $\epsilon = 10^{-5}$, and error evolution as $\epsilon \to 0$ using the discrete maximum norm $\max_{K\in\mathscr{P}_\mathcal{H}} \max_{\boldsymbol{x}\in \mathrm{DoFs}(V_h(K))} |(u - u_{Hh})(\boldsymbol{x})|$ (right).

Figure 3 displays (left) the heterogeneous permeability field $\kappa(x, y)$, (middle) the pressure field obtained with the MHM-UN method, and (right) a comparison of solution profiles at $x = 33.0$ among MHM-UN, MHM-Gal, UNUSUAL, and the reference. For this test, we use $\ell = 0$, $k = 1$, a coarse mesh with only 128 triangles, edge partition $H = \mathcal{H}/8$, and local mesh size $h = H$. Despite the coarse resolution, the MHM-UN method effectively captures the reference profile, particularly in the region near $x = 33.0$, where a sharp boundary layer appears. In contrast, MHM-Gal exhibits spurious oscillations, and the UNUSUAL method fails to resolve important features of the solution.

5 Conclusions

We proposed the MHM-UN method combining the MHM framework with stabilized local solvers based on the UNUSUAL method, extended in this work for variable coefficients. This modification improves the robustness of the multiscale basis functions in the presence of sharp boundary layers or heterogeneous coefficients allowing the use of coarse local meshes. The method is well-posed under some appropriated assumptions, and convergence estimates shows optimal rates in standard norms, with additional dependence on local Péclet numbers. Numerical results confirm that MHM-UN outperforms the MHM-Galerkin formulation in singularly perturbed problems, providing accurate and stable approximations without excessive mesh refinement.

Acknowledgements. The authors acknowledge the IPES research group (http://ipes.lncc.br/) for its support and CNPq/Brazil and CAPES for funding this research.

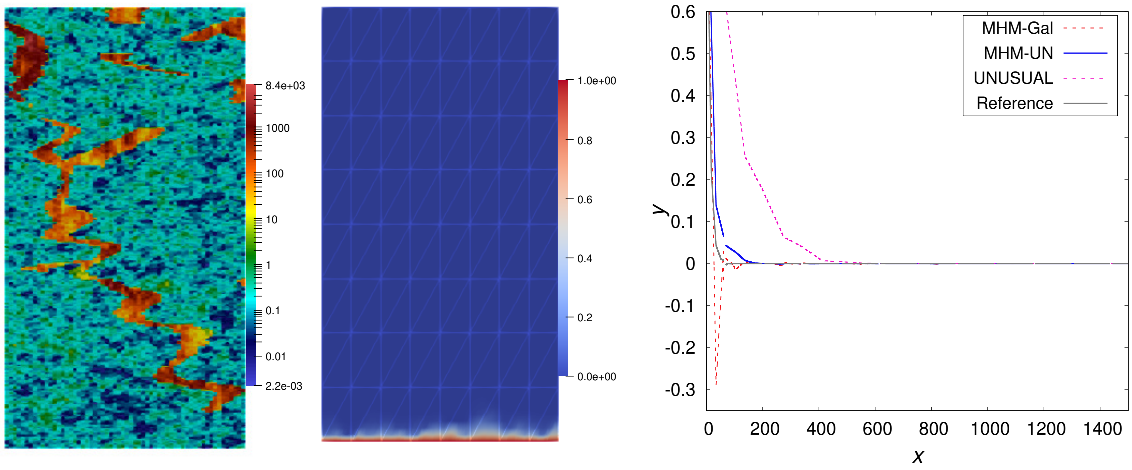

Figure 3. Layer 36 permeability field $\kappa(x, y)$ in Log-scale (left), pressure field computed with MHM-UN (middle), and profile comparison at $x = 33.0$ among MHM-UN, MHM-Gal, UNUSUAL, and the reference solution (right).

Authorship statement. The authors hereby confirm that they are the sole liable persons responsible for the authorship of this work, and that all material that has been herein included as part of the present paper is either the property (and authorship) of the authors, or has the permission of the owners to be included here.

References

[1] I. Babuška and J. E. Osborn. Generalized finite element methods: their performance and their relation to mixed methods. *SIAM Journal on Numerical Analysis*, vol. 20, n. 3, pp. 510–536, 1983.

[2] T. Hou, X.-H. Wu, and Z. Cai. Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients. *Mathematics of computation*, vol. 68, n. 227, pp. 913–943, 1999.

[3] Y. R. Efendiev, T. Y. Hou, and X.-H. Wu. Convergence of a nonconforming multiscale finite element method. *SIAM Journal on Numerical Analysis*, vol. 37, n. 3, pp. 888–910, 2000.

[4] T. J. Hughes. Multiscale phenomena: Green's functions, the dirichlet-to-neumann formulation, subgrid scale models, bubbles and the origins of stabilized methods. *Computer methods in applied mechanics and engineering*, vol. 127, n. 1-4, pp. 387–401, 1995.

[5] T. Arbogast, G. Pencheva, M. F. Wheeler, and I. Yotov. A multiscale mortar mixed finite element method. *Multiscale Modeling & Simulation*, vol. 6, n. 1, pp. 319–346, 2007.

[6] C. Harder, D. Paredes, and F. Valentin. A family of multiscale hybrid-mixed finite element methods for the darcy equation with rough coefficients. *Journal of Computational Physics*, vol. 245, pp. 107–130, 2013.

[7] R. Araya, F. Jaillet, D. Paredes, and F. Valentin. Generalizing the multiscale hybrid-mixed method for reactive-advective-diffusive equations. *Computer Methods in Applied Mechanics and Engineering*, vol. 428, pp. 117089, 2024.

[8] L. P. Franca and F. Valentin. On an improved unusual stabilized finite element method for the advective–reactive–diffusive equation. *Computer Methods in Applied Mechanics and Engineering*, vol. 190, n. 13-14, pp. 1785–1800, 2000.

[9] G. R. Barrenechea, F. Jaillet, D. Paredes, and F. Valentin. The multiscale hybrid mixed method in general polygonal meshes. *Numerische Mathematik*, vol. 145, n. 1, pp. 197–237, 2020.

[10] R. Araya, C. Harder, D. Paredes, and F. Valentin. Multiscale hybrid-mixed method. *SIAM Journal on Numerical Analysis*, vol. 51, n. 6, pp. 3505–3531, 2013.

[11] A. T. A. Gomes, W. S. Pereira, and F. Valentin. The mhm method for linear elasticity on polytopal meshes. *IMA Journal of Numerical Analysis*, vol. 43, n. 4, pp. 2265–2298, 2023.

[12] H. Fernando, L. Martins, W. Pereira, and F. Valentin. A petrov–galerkin multiscale hybrid-mixed method for the darcy equation on polytopes. *Computational and Applied Mathematics*, vol. 42, n. 4, pp. 173, 2023.

[13] A. Jaramillo, R. T. Guiraldello, S. Paz, R. F. Ausas, F. S. Sousa, F. Pereira, and G. C. Buscaglia. Towards hpc simulations of billion-cell reservoirs by multiscale mixed methods. *Computational Geosciences*, vol. 26, n. 3, pp. 481–501, 2022.