XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGINEERING

Buckling analysis of plates via the Energy Finite Difference Method (EFDM) under temperature gradients

Altamiro J. M. Silva[1], Felício B. Barros[1], Leandro L. da Silva[1]

[1]*Graduate Program in Structural Engineering – PROPEEs, Federal University of Minas Gerais - UFMG*
Av. Presidente Antônio Carlos, 6627, ZIP Code: 31270-901, Belo Horizonte, MG, Brasil
altamirojms@ufmg.edu.br, felicio@dees.ufmg.br, leandro@dees.ufmg.br

Abstract. This study aims to employ the Energy Finite Difference Method (EFDM) to analyze the buckling behavior of plates subjected to temperature gradients. The proposed methodology builds upon the classical Finite Difference Method (FDM), integrating energy principles to establish a more robust structural analysis framework. This approach enables numerical solutions with enhanced accuracy, accounting for coupled thermal and instability effects.Buckling is a critical phenomenon in two-dimensional structures, such as plates and shells, and can be significantly influenced by thermal variations, which induce material expansion and contraction. These thermal effects, combined with mechanical loading conditions, may substantially alter structural behavior.The developed method was applied to analyze plates under various thermal and mechanical loading configurations. Results were assessed through comparisons with the analytical solution, validating the results of the proposed method and the convergence of the results with the variation of parameters of the proposed examples. The findings demonstrate the efficacy of EFDM in accurately modeling buckling phenomena, highlighting its potential as a reliable tool for stability analysis of structures exposed to thermal gradients.

Keywords: structural analysis,energy finite difference method, numerical methods, plates and shells, buckling

1 Introduction

Among the elements of study in the field of structural engineering, plates, shells, and sheets stand out. The analysis of these elements allows for a clearer and more precise understanding of the physical phenomena related to them, leading to formulations that are increasingly closer to reality. For users, their study contributes to safety in use, and the cost of the structure is significantly influenced by these elements. Since they are susceptible to different situations, predicting their behavior is fundamental. Thin plates, in turn, constitute two-dimensional elements that are part of most existing structures, such as building floors, grandstands, and different types of pavements and decks. An element is classified as a plate by its dimensions, where one dimension is much smaller than the other two, with this smaller dimension being called its thickness.

According to some authors, a plate is considered thin when the ratio between its thickness and its smallest lateral dimension is less than 0,10. However, it should be noted that these values are not rules, as some other authors use ratios with different values [12]. The analysis of thin plates, with linear or nonlinear, static or dynamic behavior, is the objective of studies by several researchers, which can be verified by the number of existing publications in recent decades.

In addition to conventional loading factors, it is important to consider that plates can suffer thermal buckling, an elastic instability phenomenon that consists of the lateral deflection of the structure caused by an increase in temperature. Thermal buckling occurs when the ends of the component are prevented from moving and it is subjected to a temperature increase, which generates internal compressive stresses. A study on the numerical analysis of thermal buckling in thin steel plates, conducted by Helbig et al. [1], used a computational model based on the Finite Element Method (FEM) in ANSYS software to address this problem.

For the analysis of the behavior of thin plates, given the focus on analyzing thermal buckling, the Energy

Finite Difference Method (EFDM) will be used, which is a variation of the traditional Finite Difference Method (FDM). Among the authors with work on the subject, Girault [10], Pavlin and Perrone [11], Barve and Dey [9], Silva [5], Guang-Yao and Han-Bin [3], Fielding et al. [4], Graça [6], Mittelbach [2], and Maksimyuk et al. [8] can be cited. Girault [10] and Pavlin and Perrone [11] were among the first to present work on the numerical formulation of arbitrary meshes using finite differences. Guang-Yao and Han-Bin [3] used the Finite Difference Method in its traditional form, comparing the results with other similar works and concluding that the method is competitive with the Finite Element Method, showing high precision and lower computational processing consumption. Barve and Dey [9] incorporated the isoparametric formulation into the context of the Energy Finite Difference Method, developing an efficient and versatile tool for solving plates with curved edges.

Silva [5] presented an extension of the Energy Finite Difference Method, with the development of a flexible discretization model. The approach used allows for the solution of complex plate bending problems with curved and irregular edges, giving greater efficiency and versatility to the numerical treatment. According to Fielding et al. [4], the Energy Finite Difference Method presents versatility and simplicity for numerical solutions of problems involving plates subjected to bending. The same authors developed a model that uses a combination of Lagrange polynomial interpolation with the EFDM in arbitrary meshes, showing satisfactory precision and rapid convergence. However, in more complex irregular meshes, small distortions may arise in the results due to the non-uniformity of the distribution of nodal points. Graça [6] used the EFDM to obtain the structure's displacements, and the discretization was considered satisfactory for the calculation. Mittelbach [2] proposed the combination of Mindlin's theory with the EFDM for the solution of plate problems with axisymmetric meshes. Finally, Maksimyuk et al. [8] used a variation of the Finite Difference Method for the solution of problems of linear and nonlinear systems of plates, implementing the Kirchhoff and Timoshenko hypotheses [7].

The main objective of this study is to employ the EFDM to analyze the thermal buckling behavior of plates subjected to temperature gradients. The proposed methodology integrates energy principles into the traditional FDM, establishing a more robust structural analysis framework. This approach aims to obtain numerical solutions with enhanced accuracy and reliability for the coupled effects of instability and thermal variations

2 Materials and Methods

2.1 Utilization of the Energy Finite Difference Method

In the application of the EFDM, Kirchhoff's theory is used, in which the contribution of each integration interval to the work of internal and external forces is evaluated, as well as the membrane forces that arise due to thermal expansion and contraction. The contribution of all efforts is calculated by summing each integration section, which allows for the construction of a system of algebraic equilibrium equations that, once solved, will establish the displacements and stresses along the thin plate.

2.2 Discretization Mesh and Element Types

The contribution of each integration section of the node-based mesh will be represented by different types of elements, similar to what was established by Graça [6]. It is worth noting that in each integration element, the adopted method uses derivatives up to the second order. Each nodal point will have w as a degree of freedom. At the boundaries, a degree of freedom will be added: the rotation w'_x for points on the edges parallel to the y-axis, the rotation w'_y for points on the edges parallel to the x-axis, and for the corner nodes, both rotations. This is done to eliminate the virtual nodes introduced at the nodes located on the boundary (Fig. 1).

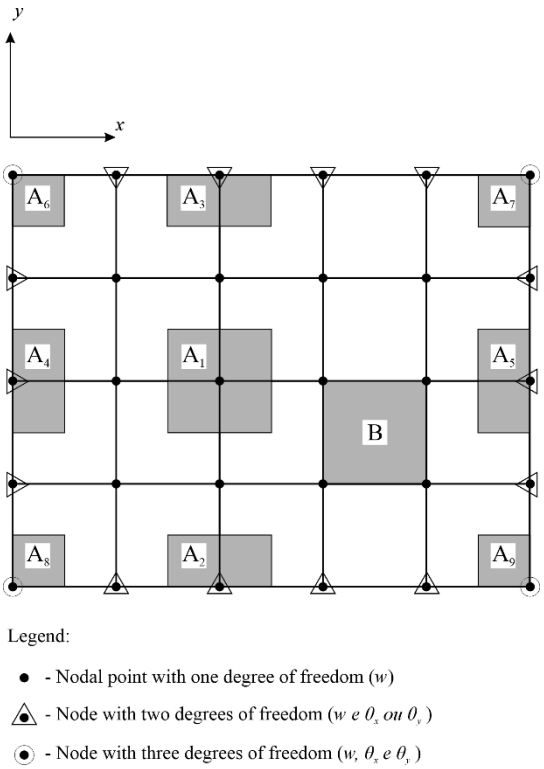


Figure 1. Integration element mesh

The mesh is discretized to apply the finite difference method, with two groups of elements: Type A elements involve direct finite differences, and Type B elements involve cross-differences.

The continuous function f(x,y) has known values at a discrete set of points (i,j). The determination of the derivatives of the function f(x,y) at the points (i,j) can be calculated by using the finite difference method.

$$\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{f_{i,j+1} - f_{i,j}}{\Delta x} \tag{1}$$

$$\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{f_{i,j} - f_{i,j-1}}{\Delta x} \tag{2}$$

$$\left(\frac{\partial f}{\partial x}\right)_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta x} \tag{3}$$

Equations 1, 2, and 3 are referred to as the forward difference, backward difference, and central difference, respectively.

According to several authors, among the finite difference expressions for approximating the first derivative of the function f(x,y) at point (i,j), the central difference generally provides a better approximation, which is illustrated by Fig. 2.

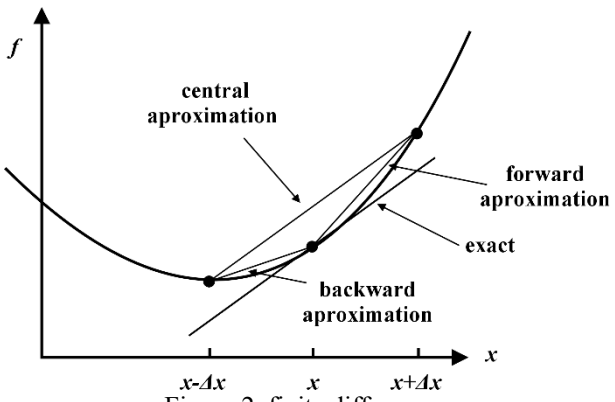


Figure 2. finite differences

Similarly to the central derivative, the expressions for the second derivative and the mixed derivative are given below in Equations 4 and 5.

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{i,j} = \frac{f_{i,j+1} - 2f_{i,j} + f_{i,j-1}}{(\Delta x)^2} \tag{4}$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{i,j} = \left(\frac{\partial^2 f}{\partial y \partial x}\right)_{i,j} = \frac{f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}}{4\Delta x \Delta y} \tag{5}$$

The partial derivative with respect to the y-axis is obtained analogously, simply by interchanging the reference axis and the corresponding spacing adopted along it.

2.3 Validation of the Computational Method

An algorithm was developed in the Maxima programming language to solve the proposed problems. For this, we used the wxMaxima interface, which is free and compatible with various operating systems, such as Windows, Linux, Mac, and Android.

The minimum critical temperature variation (ΔT_{cr}) that causes a plate, simply supported on all edges, to buckle is given by Eq. 6. This formula is derived from the theory of plate buckling subjected to uniform and isotropic compressive stresses.

$$\Delta T_{cr} = \frac{\pi^2 t^2}{12(1+\nu)\alpha}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{6}$$

where: t is the plate thickness (m); ν is Poisson's ratio; a and b are the plate dimensions (m); m and n are the numbers of half-waves formed in the longitudinal and transverse directions of the plate, respectively; α is the coefficient of thermal expansion (°C⁻¹); and π is the mathematical constant.

To validate the method, a square plate with a 1 m width was initially adopted, with a thickness of 10 mm, a Young's modulus of 200 GPa, and a coefficient of thermal expansion of 12×10^{-6} °C⁻¹, considering m and n equal to 1. From the analytical solution, a critical temperature variation of 10.54 °C is obtained. In the proposed computational method, different discretizations of the analyzed finite difference mesh were performed, yielding the results presented in Tab. 1.

Table 1. Critical Buckling Temperature Variation for a Square Plate

Mesh	ΔT_{cr}	Analytical Solution(ºC)	Δ(%)
3x3	8.43	10.54	-20.04%
6x6	10.48	10.54	-0.52%
12x12	10.60	10.54	0.55%
24x24	10.56	10.54	0.21%
48x48	10.55	10.54	0.06%

The values presented in Tab. 1 demonstrate that, even with a relatively coarse mesh discretization composed of only 6×6 divisions, it was possible to obtain a satisfactory result compared to the analytical solution, with a difference of approximately -0.52%. This result indicates that the proposed method has good accuracy even with a reduced number of elements in the mesh, which can represent a saving in computational time for more complex analyses.

Keeping the same parameters as in the previous model, but now adopting a rectangular plate with dimensions a=2.00 m and b=1.00 m, a critical temperature variation of 6.59 °C is obtained using Eq. 7. The corresponding results from the proposed computational method are presented in Tab. 2.

Table 2. Critical Buckling Temperature Variation for a Rectangular Plate

Mesh	ΔT_{cr}(Cº)	Analytical Solution (ºC)	Δ(%)
3x3	4.12	6.59	-37.48%
6x6	6.51	6.59	-1.19%
12x12	6.63	6.59	0.54%
24x24	6.60	6.59	0.21%
48x48	6.59	6.59	0.06%

As the mesh is refined, the results presented in Tab. 2 indicate that with a discretization of 12×12 divisions, it was possible to achieve very satisfactory accuracy, showing a percentage difference of approximately 0.54% compared to the analytical solution. Therefore, the results validate the proposed method for the analysis of the minimum critical temperature variation associated with the buckling of thin plates.

3 Results and Discussion

This section is dedicated to the analysis of a plate model with varying dimensions, where the width is variable and the length is fixed at 1.00 m. The following width-to-length ratios (W/L) will be evaluated: 0.25, 0.50, 0.75, 1.00, 1.25, and 1.50. Additionally, the results for different thicknesses (t=5, 10, 15 and 20mm) will be analyzed. The model has all edges clamped, a Young's modulus of 200 GPa, and a coefficient of thermal expansion of 12×10^{-6} °C^{-1}, as illustrated in Fig. 3.

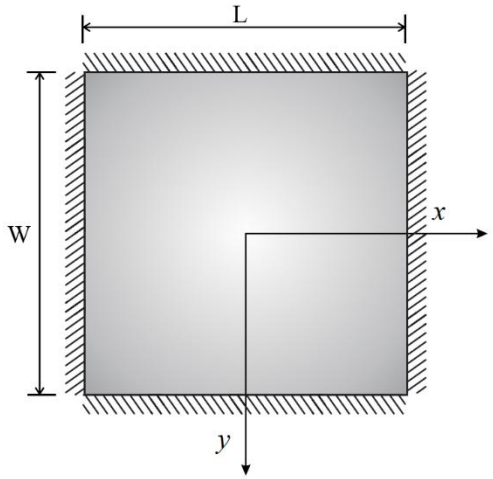


Figure 3. Rectangular plate model with four clamped edges

Fig. 4 demonstrates the convergence of the results for the case with an W/L ratio of 1.00 and a thickness of 10 mm. After 48x48 divisions in the nodal mesh, the difference between the results becomes insignificant, indicating that further refinement does not bring substantial gains in accuracy. Therefore, to obtain the data for the problem's analysis, this mesh division was adopted, ensuring an ideal balance between the fidelity of the results and computational efficiency.

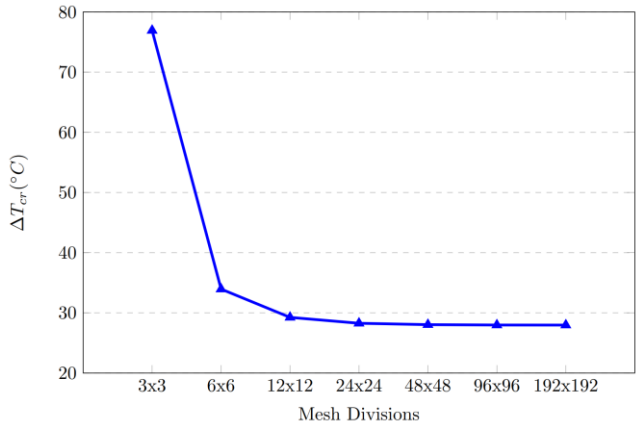


Figure 4. Convergence Analysis

To better visualize the relationship between the plate dimensions and the critical buckling temperature variation, Fig. 5 presents the results for different plate dimension ratios, considering the previously mentioned thicknesses.

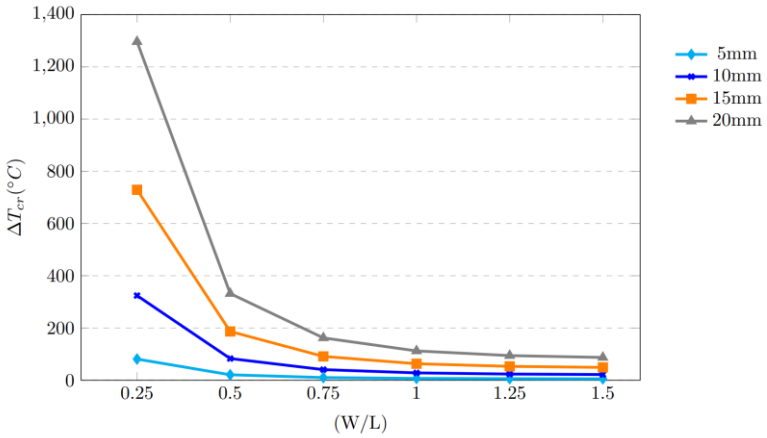


Figure 5. Temperature Variation versus W/L Ratio

As shown in Fig. 5, a smaller W/L ratio results in a significantly higher critical buckling temperature. This trend is consistently observed across all the plate thicknesses analyzed. The reason for this behavior is that with a shorter length relative to the applied load, the plate becomes stiffer and, consequently, more resistant to structural instability, requiring a greater temperature variation for buckling to occur. Conversely, as the W/L ratio increases, the plate becomes more slender and, therefore, more susceptible to buckling with smaller temperature variations.

To verify the relationship between the increase in plate thickness and the critical buckling temperature variation, Tab. 3 was created to compare the effects of this increase for the same W/L ratio.

Table 3. Critical Buckling Temperature versus Thickness

W/L	$\%\Delta T_{cr}$		
	t = 5 to 10mm	t = 10 to 15mm	t = 15 to 20mm
0.25	300.00	125.00	77.78
0.50	300.00	125.00	77.78
0.75	300.00	125.00	77.78
1.00	300.00	125.00	77.78
1.25	300.00	125.00	77.78
1.50	300.00	125.00	77.78

In Tab. 3, it is possible to note that for thinner plates, an increase in thickness results in a significant rise in the critical buckling temperature. This observation is in accordance with the principles of physics regarding the

behavior of materials under temperature variation. On the other hand, as the plate thickness increases, the effect of this increment on the critical temperature becomes less pronounced. For example, the increase in critical temperature when going from 15 mm to 20 mm is less significant than the one observed in the transition from 5 mm to 10 mm.

4 Conclusion

The Energy Finite Difference Method (EFDM) proved to be effective and reliable for the analysis of thermal buckling in rectangular plates, for both clamped and simply supported boundary conditions. The method's robustness was confirmed by comparing the result obtained with the analytical solution available in the literature, demonstrating excellent convergence and accuracy.

The developed formulation, based on the Energy Finite Difference Method, proved to be an efficient computational tool for the linear analysis of thin plates. The consistency of the results obtained validates its use as a solid alternative for studying thermal buckling problems. This approach offers a precise and flexible solution, capable of handling different geometries and boundary conditions, thus overcoming the limitations of purely analytical methods and allowing for a more in-depth investigation into the structural behavior of components subjected to high temperatures.

Acknowledgements.

The authors thank the Federal University of Minas Gerais (UFMG), the Coordination for the Improvement of Higher Education Personnel (CAPES) and the Minas Gerais Research Foundation (FAPEMIG) for their support.

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