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Boundary Element Method Applied to the Solution of Fractional Poisson Equation

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Abstract. This work presents a Boundary Element Method (BEM) formulation for the solution of the fractional Poisson differential equation, in which the order of the space derivative, say α, belongs to the interval (1,2). A comprehensive discussion regarding the development of the BEM formulation is presented and four examples are analysed. The BEM results are compared with the analytical solutions.

Keywords: boundary element method, fractional Poisson equation, fractional calculus.

1 Introduction

As old as standard calculus, the so-called fractional calculus has become, in recent decades, an emerging field of research, not only in mathematics, but in many branches of science and engineering where nonlocality plays an important role. According to Miller and Ross [1]: "the name fractional calculus has become somewhat of a misnomer. A better description might be differentiation and integration to an arbitrary order", since the theory of generalized operators has been extended to include operators of rational or irrational, positive or negative, real or complex orders. Due to tradition, and to use a shorter name, in this article the name fractional calculus will be used to refer to differential equations with non-integer operators. Today, many mathematical models that allow a better understanding of problems related to physics, chemistry, biology, rheology, finances, geophysics, fluid mechanics, ecology, and engineering, e.g., Sun et al. [2], Lischke et al. [3], Monroy and Raposo [4], are based on these differential equations. The development of such mathematical models was followed by the development of numerical methods which, in turn, thanks to their success in solving many problems, stimulated the development of new mathematical models, and so on, in a process of reciprocal conditioning. Since it is nearly impossible to provide a very complete description of the works produced so far that are concerned with numerical methods applied to the solution of fractional problems, and because even a comprehensive account of such works would require at least one article developed specifically with this purpose, the authors restricted themselves in making reference to the articles that played a fundamental role in the development of their work.

This work presents a Boundary Element Method (BEM) formulation, see Brebbia et al. [5], for the solution of the one-dimensional Fractional Poisson Equation. All the steps necessary to arrive at a typical BEM equation are presented in detail since, unlike the usual BEM formulations, instead of the unknow variable, say u, the resulting BEM equation contains space integrals of this variable. To arrive at the final system of equations, the domain is discretized with the use of linear cells.

An important aspect related to differential equations with non-integer spatial derivatives, that is, those that present a fractional Laplacian, is discussed in Lischke et al. [3]. In their comprehensive work, the authors present different definitions and interpretations for the fractional Laplacian, thus justifying its title: "What is the

fractional Laplacian?", and enriching the discussion on a subject that, according to Monroy and Raposo [4], is still an open problem.

Four examples are included in this article and, in each one, five orders of the space derivative, say α, are considered, that is: $\alpha = 1.001$, 1.2, 1.5, 1.8 and 1.999. The first and the fourth example constitute the benchmark problems in the work by Lischke et al. [3]. To verify the potentialities of the proposed formulation, BEM results are compared with the analytical solutions with good agreement observed between them, see Figures 1 – 4. Restricting the domain to the interval [–1,1], it was possible to obtain the analytical solutions by means of Jacobi polynomials, e.g. Mao and Karniadakis [6].

2 The Fractional Poisson Equation

The fractional Poisson equation is written as:

$$\frac{\partial^{\alpha} u(x)}{\partial |x|^{\alpha}} = f\left(x\right) \tag{1}$$

In eq. (1), $f(x)$, is called source function and the differential operator is such that $1 < \alpha < 2$.

If the problem domain is defined in the interval $L_1 \le x \le L_2$, then the boundary conditions are:

Dirichlet boundary condition: $u(x) = \hat{u}(x),$ at $x = L_1$ or at $x = L_2$ (2)

Neumann boundary condition: $q(x) = \dfrac{\partial u}{\partial x} = \hat{q}(x),$ at $x = L_1$ or at $x = L_2$ (3)

The Riesz space fractional derivatives of order $1 < \alpha < 2$ are defined as, see Turner et al. [7]:

$$\frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} = -c_{\alpha} \left({}_{L_1}D_x^{\alpha} + {}_{x}D_{L_2}^{\alpha} \right) u(x) \tag{4}$$

where ${}_{L_1}D_x^{\alpha}$ and ${}_{x}D_{L_2}^{\alpha}$ are defined in the sense of the left- and right-side Riemann–Liouville derivatives as:

$$\begin{aligned} {}_{L_1}D_x^{\alpha} u(x,t) &= \frac{1}{\Gamma\left(2-\alpha\right)} \frac{\partial^2}{\partial x^2} \int_{L_1}^{x} \left(x-s\right)^{1-\alpha} u\left(s\right) ds \\ {}_{x}D_{L_2}^{\alpha} u(x,t) &= \frac{1}{\Gamma\left(2-\alpha\right)} \frac{\partial^2}{\partial x^2} \int_{x}^{L_2} \left(s-x\right)^{1-\alpha} u\left(s\right) ds \end{aligned} \tag{5}$$

and

$$c_{\alpha} = \frac{1}{2\cos\left(\dfrac{\pi\alpha}{2}\right)} \tag{6}$$

In eqs. (5), $\Gamma(...)$ is the Gamma function. It is easily noticed that the classical Poisson equation can be treated as a particular case of eq. (1), in which $\alpha = 2$.

3 The Boundary Element Method Formulation

The Weighted Residuals Method (WRM), see Brebbia et al. [5], Zienckiewicz and Morgan [8], is a suitable approach for the development of the BEM formulation concerned with the solution of eq. (1). A comprehensive discussion is carried out in what follows. Regarding the BEM, the well-known fundamental solution of the steady-state classical diffusion problem, say w, is chosen as the domain weighting function, whereas the boundary weighting functions, $\overline{w}$ and $\overline{\overline{w}}$, are suitably chosen afterwards, with the aim of avoiding unnecessary approximations to the boundary conditions and, consequently, of simplifying the resulting BEM equation.

Calling *w* the fundamental solution, then it is the solution of the equation below:

$$\frac{\partial^2 w}{\partial x^2} = \delta(x-\xi) \tag{7}$$

In eq. (7), δ(…) is the Dirac delta, *x* is called the *field point* and *ξ*, the *source point*. The fundamental solution has a very simple expression, given by, see Polyanin [9]:

$$w = w(\xi, x) = \frac{|x-\xi|}{2} \tag{8}$$

For the development of the BEM formulation, and bearing in mind eq. (4) and eqs. (5), eq. (1) is initially rewritten as:

$$\frac{c_\alpha}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) + f(x) = 0 \tag{9}$$

Then, the starting, or basic, WRM equation reads:

$$\frac{c_\alpha}{\Gamma(2-\alpha)} \int_{L_1}^{L_2} \left[\frac{\partial^2}{\partial x^2} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) + f(x) \right] \bar{w}\, dx + \left(q-\hat{q}\right) \bar{w} \Big|_{x=L_1} + \left(q-\hat{q}\right) \bar{\bar{w}} \Big|_{x=L_2} = 0 \tag{10}$$

Although, for practical purposes, Neumann boundary conditions were assumed at the boundary nodes in eq. (10), that is, at $x = L_1$ and at $x = L_2$, the same final BEM equation would have been obtained for any other choice of boundary conditions. The integral that contains the second order derivative with respect to the space variable *x* is integrated by parts twice, resulting in the following expression:

$$\int_{L_1}^{L_2} \left[\frac{\partial^2}{\partial x^2} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \right] \bar{w}\, dx = \bar{w} \frac{\partial}{\partial x} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \Bigg|_{x=L_1}^{x=L_2} -$$

$$\frac{\partial \bar{w}}{\partial x} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \Bigg|_{x=L_1}^{x=L_2} + \int_{L_1}^{L_2} \left[\left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \frac{\partial^2 \bar{w}}{\partial x^2} \right] dx \tag{11}$$

Bearing in mind eq. (7), the last term on the left-hand-side of eq. (11) changes to:

$$\int_{L_1}^{L_2} \left[\left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \frac{\partial^2 \bar{w}}{\partial x^2} \right] dx = \int_{L_1}^{\xi} (\xi-s)^{1-\alpha} u(s)\, ds + \int_{\xi}^{L_2} (s-\xi)^{1-\alpha} u(s)\, ds \tag{12}$$

Substitution of eq. (12) into eq. (11), and substitution of the resulting expression into eq. (10) leads to:

$$\frac{c_\alpha}{\Gamma(2-\alpha)} \left[\int_{L_1}^{\xi} (\xi-s)^{1-\alpha} u(s)\, ds + \int_{\xi}^{L_2} (s-\xi)^{1-\alpha} u(s)\, ds \right] =$$

$$-\frac{c_\alpha}{\Gamma(2-\alpha)} \left[\bar{w} \frac{\partial}{\partial x} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \right] \Bigg|_{x=L_1}^{x=L_2}$$

$$+\frac{c_\alpha}{\Gamma(2-\alpha)} \left[\frac{\partial \bar{w}}{\partial x} \left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right) \right] \Bigg|_{x=L_1}^{x=L_2}$$

$$-\left(q-\hat{q}\right) \bar{w} \Big|_{x=L_1} + \left(q-\hat{q}\right) \bar{\bar{w}} \Big|_{x=L_2} + \int_{L_1}^{L_2} f\, \bar{w}\, dx \tag{13}$$

The choice of weighting functions $\bar{w}$ and $\bar{\bar{w}}$ is required to obtain the final form of a rather new BEM equation which, instead of being written for *u*(ξ), is written for ${}_{L_1}D_{\xi}^{\alpha} u(\xi,t)$ and ${}_{\xi}D_{L_2}^{\alpha}(\xi,t)$. The computation of

the derivative, with respect to *x*, of the first integrals on the right-hand-side of eq. (13) will enable one to make the proper choice of the weighting functions $\overline{w}$ and $\overline{\overline{w}}$. However, to succeed in this attempt it becomes necessary to discretize the interval $[L_1, L_2]$ into cells. If *N* linear cells of the same length, say $\Delta L = (L_2 - L_1)/N$, are employed, then one has: $x_j = L_1 + j\Delta L$, $j = 0,1,2,...,N$.

For $x = L_1$:

$$\frac{\partial}{\partial x}\left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right)\Bigg|_{x=L_1} =$$

$$(1-\alpha)\int_{L_1}^{L_2} (s-L_1)^{-\alpha} u(s)\, ds + q(L_1) \int_{L_1}^{L_1+\Delta L} (s-L_1)^{1-\alpha} \varphi_{L_1}(s)\, ds = (1-\alpha)\int_{L_1}^{L_2} (s-L_1)^{-\alpha} u(s)\, ds + \kappa q(L_1) \tag{14}$$

For $x = L_2$:

$$\frac{\partial}{\partial x}\left(\int_{L_1}^{x} (x-s)^{1-\alpha} u(s)\, ds + \int_{x}^{L_2} (s-x)^{1-\alpha} u(s)\, ds \right)\Bigg|_{x=L_2} =$$

$$(1-\alpha)\int_{L_1}^{L_2} (L_2-s)^{-\alpha} u(s)\, ds + q(L_2) \int_{L_2-\Delta L}^{L_2} (x-s)^{1-\alpha} \varphi_{L_2}(s)\, ds = (1-\alpha)\int_{L_1}^{L_2} (L_2-s)^{-\alpha} u(s)\, ds + \kappa q(L_2) \tag{15}$$

The linear interpolation functions φ_{L_1} and φ_{L_2} in eqs. (14 – 15), respectively, are defined according to:

$$\varphi_{L_1}(s) = \frac{(L_1 + \Delta L) - s}{\Delta L} \tag{16}$$

$$\varphi_{L_2}(s) = \frac{s - (L_2 - \Delta L)}{\Delta L} \tag{17}$$

The constant κ, in eqs. (14 – 15), is given by:

$$\kappa = \frac{(\Delta L)^{2-\alpha}}{(2-\alpha)(3-\alpha)} \tag{18}$$

The weighting functions $\overline{w}$ and $\overline{\overline{w}}$ are determined by assuming that there will be no contributions from the approximate values, $q(L_2)$ and $q(L_1)$, in the BEM equation, which means that:

$$\frac{c_\alpha}{\Gamma(2-\alpha)} w(\xi, L_1)\kappa q(L_1) - q(L_1)\overline{w} = 0 \tag{19}$$

$$-\frac{c_\alpha}{\Gamma(2-\alpha)} w(\xi, L_2)\kappa q(L_2) - q(L_2)\overline{\overline{w}} = 0 \tag{20}$$

From (19) and (20) one has:

$$\overline{w} = \frac{c_\alpha}{\Gamma(2-\alpha)} w(\xi, L_1)\kappa \tag{21}$$

$$\overline{\overline{w}} = -\frac{c_\alpha}{\Gamma(2-\alpha)} w(\xi, L_2)\kappa \tag{22}$$

Considering all the steps taken previously, the BEM equation is finally obtained; it reads:

$$\frac{c_\alpha}{\Gamma(2-\alpha)}\left[\int_{L_1}^{\xi} (\xi - s)^{1-\alpha} u(s)\, ds + \int_{\xi}^{L_2} (s-\xi)^{1-\alpha} u(s)\, ds \right] =$$

$$-\frac{c_\alpha}{\Gamma(2-\alpha)}\left[w(\xi,L_2)(1-\alpha)\int_{L_1}^{L_2}(x-s)^{-\alpha}u(s)ds+w(\xi,L_1)(1-\alpha)\int_{L_1}^{L_2}(s-x)^{-\alpha}u(s)ds\right]$$

$$+\frac{c_\alpha}{\Gamma(2-\alpha)}\left[\frac{1}{2}\int_{L_1}^{L_2}(x-s)^{1-\alpha}u(s)ds+\frac{1}{2}\int_{L_1}^{L_2}(s-x)^{1-\alpha}u(s)ds\right]$$

$$-\frac{c_\alpha}{\Gamma(2-\alpha)}w(\xi,L_2)\kappa q(L_2)+\frac{c_\alpha}{\Gamma(2-\alpha)}w(\xi,L_1)\kappa q(L_1)\ -\left(\int_{L_1}^{\xi}\frac{(\xi-x)}{2}f(x)dx+\int_{\xi}^{L_2}\frac{(x-\xi)}{2}f(x)dx\right) \tag{23}$$

The domain integral containing the function $f(x)$ is easily evaluated, once again using linear cells. Writing eq. (23) for each $\xi_j = L_1 + j\Delta L$, $j = 0,1,2,...,N$, results in a system of equations, whose matrix representation is:

$$\boldsymbol{Hu} = \boldsymbol{Gq} - \boldsymbol{n} \tag{24}$$

where matrix $\boldsymbol{H}$ is given by:

$$\boldsymbol{H} = \bar{\boldsymbol{H}} - \boldsymbol{H}_w - \boldsymbol{H}_{1/2} \tag{25}$$

In eq. (25) matrices $\bar{\boldsymbol{H}}$, $\boldsymbol{H}_w$ and $\boldsymbol{H}_{1/2}$ are related, respectively, to the integrals on the left-hand-side and to the first and second integrals on the right-hand-side in eq. (23). Regarding eq. (24), matrix $\boldsymbol{G}$ comes from the third and fourth terms in eq. (23) and the vector $\boldsymbol{n}$ results from the domain integrals of the function $f(x)$. The solution contains the values of u, or q, at the boundary nodes and the values of u at the internal points.

4 Examples

To validate the BEM results, four examples are presented in this section, with Dirichlet boundary conditions at $L_1 = -1$ and at $L_2 = 1$, for $\alpha = 1.001$; 1.2; 1.5; 1.8; 1.999. In all examples the BEM results are compared with the analytical solution; anticipating what the following figures show, good agreement is always observed between them. BEM analyses employed a mesh constituted by *ncell* = 128 linear cells of the same size. To arrive at the conclusion that this number of cells is adequate to produce accurate results, the third example, with $\alpha = 1.001$, was carefully investigated (preliminary results are not included here due to lack of space). After several attempts, the authors arrived at the conclusion that *ncell* = 128 was a good choice, that is, it could provide accurate results.

Before presenting the BEM results, it is important to mention that Jacobi polynomials, $P_n(a,b,x)$, are employed as convenient basis functions for the series expansion of the functions $u(x)$ and $f(x)$. The interested reader is referred to Mao and Karniadakis [6] for additional information regarding this matter.

The function $f(x)$, in the first example, is: $f(x) = -P_0$, $P_0 = 1$ is the Jacobi polynomial of order zero; in the second example, $f(x) = -P_1$, $P_1 = (2+\alpha)x/2$ is the Jacobi polynomial of order one. Note that due to the orthogonality of the Jacobi polynomials, in the series expansion for obtaining the analytical solution only one of the factors is not null and, consequently, the analytical solution is given by a single expression. For the first example, it reads:

$$u(x) = \frac{(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\alpha}{2}}}{\Gamma(1+\alpha)} \tag{26}$$

For the second example, one has:

$$u(x) = \frac{(2+\alpha)x(1-x)^{\frac{\alpha}{2}}(1+x)^{\frac{\alpha}{2}}}{2\Gamma(1+\alpha)} \tag{27}$$

Figures 1 – 4 below depict the results corresponding to the forcing functions: *i)* $f(x) = -P_0$, *ii)* $f(x) = -P_1$, *iii)* $f(x) = -x^2$, *iv)* $f(x) = -\sin(\pi x)$.

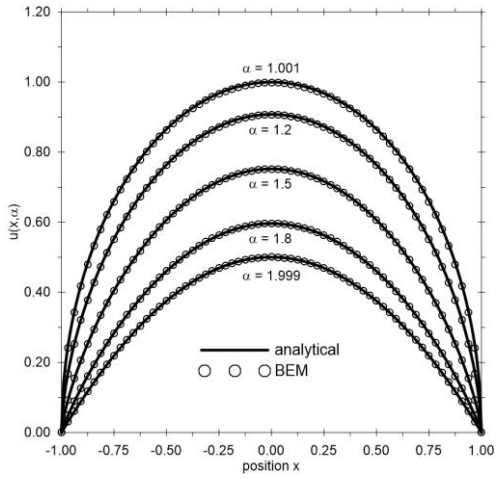


Figure 1. BEM results and analytical solution for the source function $f(x) = -P_0$.

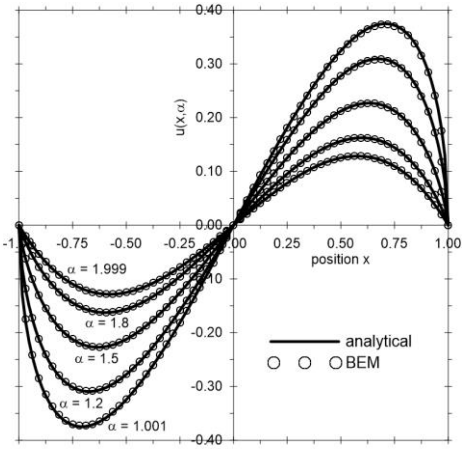


Figure 2. BEM results and analytical solution for the source function $f(x) = -P_1$.

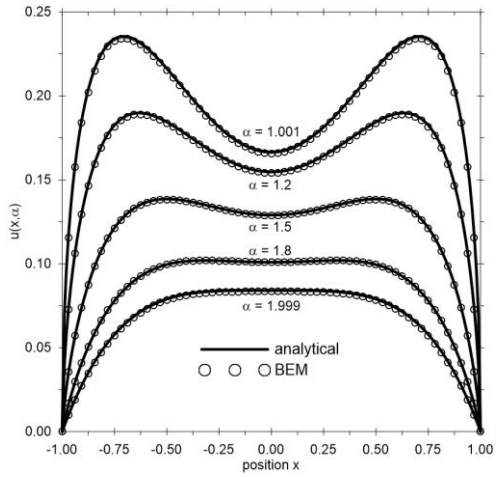


Figure 3. BEM results and analytical solution for the source function $f(x) = -x^2$.

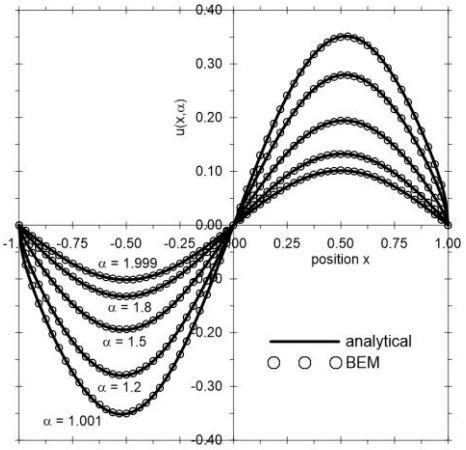


Figure 4. BEM results and analytical solution for the source function $f(x) = -\sin(\pi x)$.

5 Conclusions

Solving differential equations, both ordinary and partial, with non-integer derivatives has proven to be a challenge for interested researchers. Equations containing only non-integer time derivatives have been successfully solved: the interested reader will certainly find many articles related to this subject, among which those recently published by the authors of the present article. On the other hand, the researcher is faced with even greater difficulties when trying to solve problems with non-integer spatial derivatives: the authors can confirm this observation, since this article began to be conceived a long time ago and could only recently be completed. Without any hesitation in saying so, it was successfully completed as demonstrated by the comparison between BEM results and the analytical solutions, obtained via Jacobi Polynomials. For the future, new steps must be taken which include the solution of the one-dimensional anomalous diffusion problem, with non-integer derivatives in time.

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