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A Geometrically-Exact Nonlinear Shell Formulation Based on Efficient Triangular Elements

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Abstract. Thin-walled structures are part of many engineering applications, as wind turbine blades, pressure vessels and sheet metal products. A thin-walled structure is characterized by one significantly smaller dimension compared to the others. To numerically support the designing process of such industrial structures, a robust and efficient approximation algorithm is required. This work aims to examine the behavior of a nonlinear formulation for shell models using an efficient triangular shell element. Thereby, the proposed displacement-based triangular shell element, which comprises six nodes, has a compatible linear interpolation scheme for displacements and a non-conforming linear rotation field, resulting in a very efficient finite element formulation. This particular formulation is based on Reissner-Mindlin kinematic assumptions and an initial plane reference configuration for the shell. The proposed formulation also accounts for finite strains, large displacements, and rotations. Additionally, the rotation field has been re-parameterized using the Rodrigues rotation parameters, facilitating an efficient update of the rotational field in comparison to the classical Euler rotation vector.

Keywords: Nonlinear Shell Formulation, Rodrigues Rotation Parameters, Triangular Elements, Thin-Walled Structures

1 Introduction

In solid mechanics, shell elements are classified by the associated kinematic model. A significant group of elements are based on the Reissner-Mindlin hypotheses. Reissner [1] and Mindlin [2] both developed structural models for moderately thick plates that incorporated transverse shear and allowed for distortion of the cross-sections. These models also allowed for the fact that, although the cross-sections were flat, they do not remain orthogonal to their deformed mid-surface, similar to the assumptions in the Timoshenko beam theory. In Mindlin's formulation, this leads to a uniform shear distribution through the plate thickness, necessitating a shear correction factor to adjust the stress field to satisfy natural boundary conditions. The shell model developed by Campello [3], known as T6-3i, is based on the Reissner-Mindlin structural theory. It is geometrically exact and applicable for large deformations. This element explicitly handles finite rotations using the Euler-Rodrigues tensor, ensuring that the cross-sections are accurately represented. It is also referred to as shear flexible element, as it is capable of handling shear strain within their kinematic description. The kinematic assumptions underlying this model are adapted in the development of the shell element proposed in this work. One of the major challenges in developing geometrically exact shell theories has been the consistent description of finite rotations. Argyris established in [4] a milestone in this approach, but there were still leaving open questions. To address this, various parameterizations were developed, most notably the models based on the rotation vector and the Euler-Rodrigues tensor [3] and the model of the first consistent linearization of this parameterization developed by Pimenta and Yojo [5]. Shells models incorporating the Rodrigues rotations vector as parametrization of the rotational field is provided in [6]. The rotation is described as consecutive updates using incremental parameters. In the context of shell structures, nonconforming elements are particularly useful for modeling geometric discontinuities such as holes, gaps, or intersecting surfaces, situations where conforming elements may struggle to provide accurate representations. Triangular elements are especially advantageous in nonconforming or irregular meshes, offering flexibility in handling complex geometries. Therefore, this work deals with a nonlinear formulation for shells using a nonconforming triangular

shell element. The formulation is based on the Reissner–Mindlin kinematic assumptions and an initial planar reference configuration. Furthermore, a re-parameterization of the rotation field is introduced using the Rodrigues rotation vector [9, 10]. The rotational variables are updated through an efficient incremental scheme employing the Rodrigues expression for superposed rotations (see [4, 11]).

2 Shell Kinematics and Finite Element Formulation

This section briefly presents the kinematic assumptions and finite element formulation adopted for the shell element developed in this work. The approach is grounded in a geometrically nonlinear framework, where the shell behavior is described relative to its initial reference configuration, oriented by a director vector.
The shell element formulation is based on a kinematic description relative to the initial reference configuration. In this configuration, the discretization is defined with respect to the shell's middle surface, which is assumed to be planar. An orthonormal basis system $\{\mathbf{e}_1^r, \mathbf{e}_2^r, \mathbf{e}_3^r\}$ is introduced, where $\mathbf{e}_1^r$ and $\mathbf{e}_2^r$ lie on the reference mid-surface and $\mathbf{e}_3^r$ is normal to it. In the current configuration, the shell is described by a local orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The director vectors in both configuration are defined as

$$\boldsymbol{a}^r = \boldsymbol{\zeta}\ \mathbf{e}_3^r \quad \text{and} \quad \boldsymbol{a} = \mathbf{Q}\boldsymbol{a}^r, \tag{1}$$

where $\boldsymbol{\zeta}$ denotes the position vector of a point on the reference mid-surface, and the director vector at the current configuration evolves according to the rotation tensor $\mathbf{Q}$. This formulation assumes that the director remains undeformed during shell motion, undergoing only rigid-body rotation. Thickness changes are neglected, and first-order shear deformations are accounted for, in accordance with the classical Reissner-Mindlin hypothesis [1, 2]. Consequently, the degrees of freedom in the shell model consist of the displacement components of points on the reference mid-surface and the components of the rotation vector, all expressed in the global Cartesian coordinate system. To describe the rotation tensor $\mathbf{Q}$, the Rodrigues parametrization is adopted, following the works of Pimenta et al. [6, 9]. In this approach, the rotation vector $\boldsymbol{\alpha}$ is defined as

$$\boldsymbol{\alpha} = \alpha\mathbf{e} \quad \text{where} \quad \alpha = 2\tan\frac{\theta}{2} \quad \text{with} \quad 0 \leq \theta < \pi, \tag{2}$$

where $\alpha = \|\boldsymbol{\alpha}\|$ and θ is the rotation angle. The rotation tensor is then expressed in terms of the skew-symmetric matrix $\boldsymbol{A} = \text{skew}(\boldsymbol{\alpha})$ as

$$\mathbf{Q} = \mathbf{I} + \frac{4}{4+\alpha^2}\left(\mathbf{A} + \frac{1}{2}\mathbf{A}^2\right) \quad \text{with} \quad \mathbf{A} = \text{skew}(\boldsymbol{\alpha}). \tag{3}$$

This parametrization offers computational efficiency due to its compact representation. However, it introduces singularities near $\theta = \pi$, which are circumvented by implementing the rotation update in incremental steps. It is important to note that this incremental scheme is specific to the Rodrigues parametrization and is not applicable to Euler parametrization. The strain measures in the shell are defined with respect to the deformation gradient at the reference configuration, while the stress measures are based on the first Piola-Kirchhoff stress tensor, this tensors are energetically conjugate. The local equilibrium of the shell, including essential boundary conditions, is derived in the weak form using the principle of virtual work, which serves as the foundation for the finite element implementation. To enhance the constitutive modeling, a slight modification of the deformation gradient is introduced to allow transverse (through-the-thickness) strains. Specifically, the strain component γ_{33} is incorporated as an additional degree of freedom. This component is subsequently eliminated at the constitutive level by enforcing a plane-stress condition, thereby maintaining the physical consistency of the shell formulation. A neo-Hookean material model is adopted in this study, for which the specific energy function is analytically derived. This choice provides a robust and efficient framework for capturing nonlinear material behavior.

3 Numerical Example

In this work, we consider a new triangular shell element with six nodes that is planar in the reference configuration. The element features three nodes with displacement degrees-of-freedom positioned at the vertices, and three nodes

with rotational degrees-of-freedom located at the mid-side positions. The interpolation scheme employed is linear for both displacement and rotation fields.
The proposed element is applied to test beam bending under large displacements and rotations. The cantilever beam used in the analysis has a square cross-section with dimensions b = h = 0.1 m, a length L = 1 m, and material properties E = 10^7 Pa and ν = 0.3. A concentrated point load of F = 1000 N is applied at the center of the beam's free end in two distinct directions: (a) along the beam's plane to induce in-plane bending, and (b) along the plane of the beam's thickness to simulate out-of-plane bending, as proposed by [3, 12–14] (see Fig 1(a)).

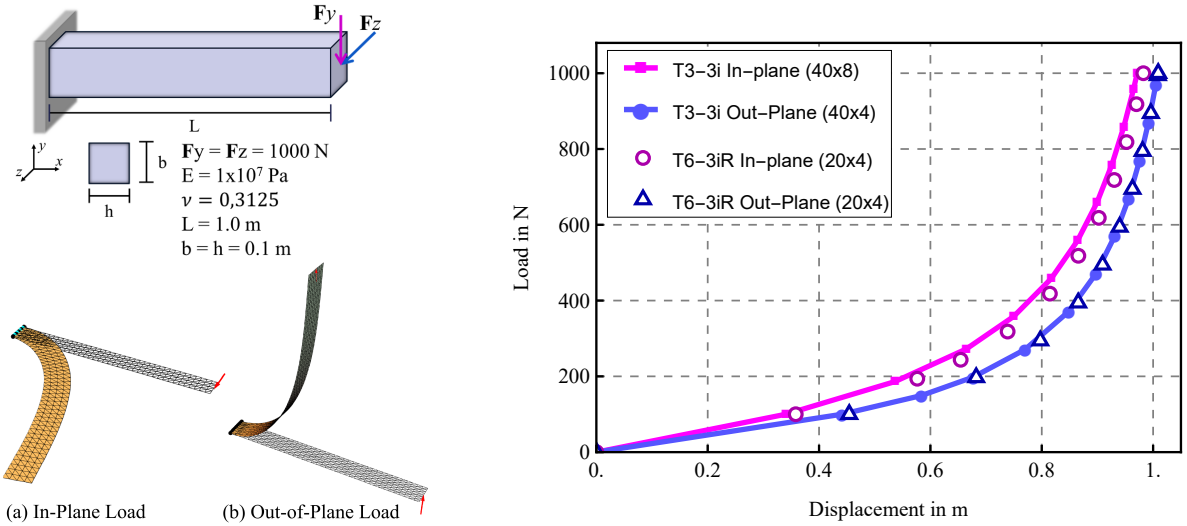


Figure 1. Geometry and boundary conditions of the cantilever beam used in the simulations and load–displacement curve and deformed mesh configurations. The top left image illustrates the beam configuration, while the bottom left shows de deformed configuration for the load of F = 1000 N in two loading scenarios: (a) in-plane load direction and (b) out-of-plane load direction. The right image shows the lines correspond specifically to the T3-3i element and the open markers correspond to the T6-3iR element.

The resulting load-deflection curves of the T3-3i element are presented in Fig. 1 correspond to the converged results for both loading scenarios, the in-plane direction, capturing membrane behavior, and the out-of-plane direction, illustrating plate bending behavior. For comparative purposes, the figures also include solutions from the T6-3iR element, a modified version of T6-3i element presented in [3], designated here as T6-3iR (Rodrigues based). Also, this version incorporates the same incremental Rodrigues rotation formulation used for the T3-3i element. As illustrated in Fig. 1, under in-plane loading, a comparison between the converged results of the T3-3i and T6-3i (Rodrigues based) elements revealed similar results. Regarding the converged results for the out-of-plane load, the T3-3i and T6-3i (Rodrigues based) elements displayed identical behavior, confirming the consistency of the formulation.

4 Conclusions

In this work, the applicability of the proposed element is demonstrated through numerical examples that highlight its promising results. Given its linear interpolation and six degrees of freedom (DOFs), the element proves capable of undergoing large displacements and rotations, capturing both membrane and plate behavior.

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