XLVI CILAMCE
VITÓRIA - ES - BRAZIL - 2025
XLVI IBERO-LATIN AMERICAN CONGRESS ON COMPUTATIONAL METHODS IN ENGENEERING

On the non-linear vibrations of fluid-filled clamped-free cylindrical shells subjected to harmonic loads.

José Eduardo Follana Peralta[1], Zenón José Guzmán Nuñez Del Prado[1], Frederico Martins Alves Da Silva[1]

[1]*School of Civil and Environmental Engineering, Federal University of Goiás*
Av. Universitária, 1488, St. Universitário,74605-220, Goiânia, GO, Brazil
jose.follana@discente.ufg.br, zenon@ufg.br, silvafma@ufg.br

Abstract. This study addresses the nonlinear vibrations of partially fluid-filled clamped-free cylindrical shells subjected to two harmonic excitations: a harmonic lateral point load and a harmonic base excitation. The fluid is considered incompressible and inviscid, and the internal filling level is maintained partially filled to evaluate its influence on the structural response. In such configurations, fluid-structure interaction becomes a critical aspect in the design and performance of these structural systems. The shell is modeled using the nonlinear Sanders-Koiter theory, with discretization based on the shell's natural modes and the derivation of the governing equations through the Lagrangian energy approach. To obtain the nonlinear dynamic equations, Hamilton's principle is applied, which are in turn solved by the fourth-order Runge-Kutta method. The displacement fields are expanded using only trigonometric functions, ensuring the exact satisfaction of boundary conditions and promoting rapid modal convergence. Five different modal expansions, ranging from 15 to 48 generalized coordinates, are employed to investigate the convergence of the solution. The lowest natural frequency of the system is computed and compared against analytical solutions reported in the literature. The results reveal that, depending on the shell's geometric parameters and the partial fluid level, different patterns of natural frequencies emerge along with complex nonlinear dynamic behavior.

Keywords: cylindrical shells, nonlinear vibrations, fluid-structure interaction.

1 Introduction

Due to their high capacity to withstand internal and external loads, circular cylindrical shells are widely employed in various branches of engineering. Their dynamic behavior is of particular interest in applications involving fluid–structure interaction, such as water reservoirs, pipelines, and fluid transport systems. The literature contains numerous studies on the linear and nonlinear analysis of simply supported cylindrical shells; however, investigations involving clamped-free boundary conditions are relatively limited. While the linear vibrations of clamped-free cylindrical shells have been addressed in earlier works Warburton and Higgs [1], Sharma [2], Tottenham and Shimizu [3], studies focusing on their nonlinear vibrations remain scarce. Among the most notable contributions is the experimental research by Chiba [4, 5], who analyzed the nonlinear free vibrations of polyester shells. Similarly, Kurylov and Amabili [6] investigated the nonlinear forced vibrations of clamped-free cylindrical shells using Chebyshev polynomials to represent the displacement fields. More recently, Montes [7] examined both free and forced vibrations in cylindrical shells with internal and external fluid. A major challenge in these studies is the proper selection of modal expansions to discretize the displacement field, particularly in nonlinear analysis. In the present work, the nonlinear vibrations of clamped-free cylindrical shells are studied, considering the presence of partially filled internal fluid. The structures are subjected to two types of dynamic excitation: a concentrated harmonic lateral load applied at midspan, and a harmonic base excitation. The nonlinear shell theory of Koiter–Sanders is adopted to model the shell behavior, and the Rayleigh–Ritz method is employed to derive a set of nonlinear dynamic equations. These equations are then solved using the fourth-order Runge–Kutta method. To accurately capture modal coupling effects, a modal expansion with fifteen degrees of freedom is considered in the axial, radial, and circumferential directions.

2 Mathematical Formulation

Consider a perfect clamped-free elastic cylindrical shell with radius R, thickness h, length L, density ρ, Young's modulus E, and Poisson's ratio ν, subjected to a harmonic base excitation $A_g(t) = y_0 \cos(\omega_b t)$ and a radial harmonic load $P(t) = \tilde{f} \cos(\omega_b t)$ acting at a height of $L/2$, as illustrated in Fig. 1, where y_0 is the amplitude of the base displacement and $\tilde{f}$ is the amplitude of the lateral point load applied to the shell. The displacement field of the middle surface of the shell consists of axial $u(x, \theta, t)$, circumferential $v(x, \theta, t)$, and radial $w(x, \theta, t)$ components, where x and θ are the axial and circumferential coordinates, and t denotes time. The shell is partially filled with an incompressible fluid of density ρ_f, reaching a height H_i from the base of the cylinder.

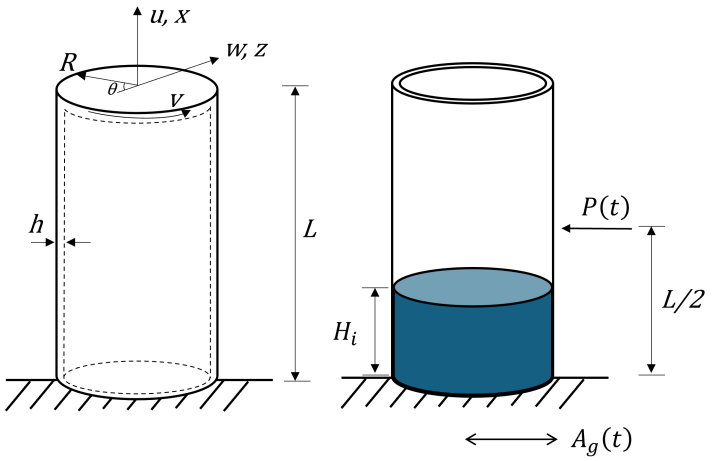


Figure 1. Geometry and coordinates of cylindrical shell

The strain relations of the cylindrical shell can be written as $\varepsilon_x = \varepsilon_{x,0} + z\kappa_x$, $\varepsilon_\theta = \varepsilon_{\theta,0} + z\kappa_\theta$, $\gamma_{x\theta} = \gamma_{x\theta,0} + z\kappa_{x\theta}$, where z is the distance of the arbitrary point of the shell from the middle surface. The middle surface strain–displacement relations, changes of curvature and torsion according to Sanders–Koiter nonlinear shell theory are given by:

$$\varepsilon_{x,0} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{8}\left(\frac{\partial v}{\partial x} - \frac{1}{R}\frac{\partial u}{\partial \theta}\right)^2 \tag{1}$$

$$\varepsilon_{\theta,0} = \frac{1}{R}\frac{\partial v}{\partial \theta} + \frac{w}{R} + \frac{1}{2}\left(\frac{1}{R}\frac{\partial w}{\partial \theta} - \frac{v}{R}\right)^2 + \frac{1}{8}\left(\frac{1}{R}\frac{\partial u}{\partial \theta} - \frac{\partial v}{\partial x}\right)^2 \tag{2}$$

$$\gamma_{x\theta,0} = \frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\left(\frac{1}{R}\frac{\partial w}{\partial \theta} - \frac{v}{R}\right) \tag{3}$$

$$\kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_\theta = \frac{1}{R^2}\frac{\partial v}{\partial \theta} - \frac{1}{R^2}\frac{\partial^2 w}{\partial \theta^2}, \quad \kappa_{x\theta} = -\frac{2}{R}\frac{\partial^2 w}{\partial x \partial \theta} + \frac{1}{2R}\left(3\frac{\partial v}{\partial x} - \frac{1}{R}\frac{\partial u}{\partial \theta}\right) \tag{4}$$

The elastic strain energy, neglecting σ_z for plane stress, is given by:

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{L}\int_{-h/2}^{h/2} (\sigma_x \varepsilon_x + \sigma_\theta \varepsilon_\theta + \tau_{x\theta}\gamma_{x\theta})\, dx(1 + \frac{z}{R}) d\theta R dz \tag{5}$$

where stress–strain relations are given by:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_x), \quad \tau_{x\theta} = \frac{E}{2(1+\nu)}\gamma_{x\theta} \tag{6}$$

The kinetic energy, considering only translational inertia, is given by:

$$T = \frac{1}{2}\rho h \int_0^{2\pi}\int_0^L \{\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) + (\dot{A}_g(t)^2 + 2\dot{A}_g(t)[\dot{w}\cos(n\theta) - \dot{v}\sin(n\theta)])\} R dx d\theta \tag{7}$$

The work of external forces is:

$$W = \int_0^{2\pi}\int_0^L \{P(t)[\delta(\theta - 0)\delta(x - \frac{L}{2})] - p_{Hi}[H_f(x) - H_f(x - H_i)]\} w R dx d\theta \tag{8}$$

where δ and H_f are the Dirac and Heaviside functions, respectively, and p_{Hi} is the hydrodynamic pressure. The funcion p_{Hi} is defined as:

$$p_{Hi} = -\rho_f \left[n + \lambda \frac{I_{n-1}(\lambda)}{I_n(\lambda)}\right]^{-1} \left(\frac{\partial^2 w}{\partial t^2}\right) \tag{9}$$

with

$$\lambda = \left(\frac{m\pi L}{R}\right) \tag{10}$$

where ρ_f is the fluid density, n is the number of circumferential waves of the cylindrical shell, and I_n and I_{n-1} are the modified Bessel functions of the first kind. Finally, the work of nonconservative forces is:

$$F = \frac{1}{2} c \int_0^{2\pi}\int_0^L \left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right) dx R d\theta \tag{11}$$

where c is the viscous damping coefficient. The field displacements can be expanded as:

$$\begin{aligned}
w(x,\theta,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N} h\, O_1(m,x)\left[W_{m,n}(t)\cos(n\theta) + W_{m,n,c}(t)\sin(n\theta)\right], \\
u(x,\theta,t) &= \sum_{m=1}^{M}\sum_{n=1}^{N} h\, O_2(m,x)\left[U_{m,n}(t)\cos(n\theta) + U_{m,n,c}(t)\sin(n\theta)\right], \\
v(x,\theta,t) &= \sum_{m=1}^{M}\sum_{n=0}^{N} h\, O_1(m,x)\left[V_{m,n}(t)\sin(n\theta) + V_{m,n,c}(t)\cos(n\theta)\right]
\end{aligned} \tag{12}$$

with

$$O_1(m,x) = 1 - \cos\left[\frac{1}{2L}(2m-1)\pi x\right], \quad O_2(m,x) = \sin\left[\frac{1}{2L}(2m-1)\pi x\right] \tag{13}$$

where m is the axial wave number, n is the circumferential wave number, subscript c refers to companion mode. The boundary conditions of a clamped-free cylindrical shell are given by:

$$u = v = w = \frac{\partial w}{\partial x} = 0 \text{ at } x = 0, \quad N_x = N_{x\theta} + \frac{M_{x\theta}}{R} = M_x = Q_x + \frac{\partial M_{x\theta}}{R\partial\theta} = 0 \text{ at } x = L \tag{14}$$

Substituting the field displacements of Eq. (10) into Eq. (14), the set of nonlinear equations of dynamic equilibrium can be obtained:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial U}{\partial q_i} = -\frac{\partial F}{\partial \dot{q}_i} + \frac{\partial W}{\partial q_i}, \quad i = 1, \ldots, \text{DOF} \tag{15}$$

where $q_i = [U_{1,0}, U_{2,0}, \ldots, U_{1,n}, U_{2,n}, \ldots, V_{1,0}, V_{2,0}, \ldots, V_{1,n}, V_{2,n}, \ldots, W_{1,0}, W_{2,0}, \ldots, W_{1,n}, W_{2,n}, \ldots]$.

3 Numerical Results.

A clamped-free cylindrical shell geometry, with Young's modulus $E = 4.65 \times 10^9$ N/m^2, density $\rho = 1400$ kg/m^3, and Poisson's ratio $\nu = 0.38$ was considered. Table 1 presents the selected shell geometry, as well as the corresponding L/R and R/h ratios. First, in order to obtain the natural frequency, a linear free vibrations analysis was performed considering the following field displacements (without companion mode) $u = (1,0)+(2,0)+(3,0)+\cdots+(1,n)+(2,n)+(3,n)+\cdots$, $v = (1,0)+(2,0)+(3,0)+\cdots+(1,n)+(2,n)+(3,n)+\cdots$ and $w = (1,0) + (2,0) + (3,0) + \cdots + (1,n) + (2,n) + (3,n) + \cdots$. Table 1 also shows the natural frequency and corresponding m, n for the shell.

Table 1. Geometric and dynamic properties of the clamped-free cylindrical shells

L (m)	R (m)	h (mm)	L/R	R/h	ω_0 (rad/s)	(m,n)
0.48	0.24	0.254	2.0	944	190.125	(1,7)

For the nonlinear analysis of the shell, the correct displacement fields that exhibit coupling and modal interaction with softening behavior must be considered. For this purpose, five displacement fields without considering the companion mode were selected and are shown in Table 2. As can be seen, displacement fields with fifteen, seventeen, eighteen, nineteen, and forty-three degrees of freedom were selected, containing symmetrical and axisymmetrical modes.

In Fig. 2, the normalized frequency–amplitude curves for the cylindrical shell, obtained with each modal expansion, are presented. All curves exhibit softening behavior. To determine which expansion best reproduces the nonlinear response, they were compared with the experimental results of Chiba [4, 5]. From this comparison, expansion $15a$ is identified as the closest match to the experimental data, and will therefore be employed in the nonlinear vibration analysis both with and without fluid.

Now, the nonlinear forced vibration analysis is considered. In this case, the shell is subjected only to a lateral harmonic point load $P(t)$, applied at the mid-span of the structure, acting when $x = L/2$, without internal fluid. A viscous damping coefficient of $\zeta = 0.001$ is adopted for the analysis. A nondimensional excitation in lateral direction $\tilde{f}$ is introducing , witch is related to the dimensional force excitation by $f_1 = \tilde{f}/m_i\omega_0^2 h$, ω_0 where is the fundamental natural frequency in rad/s and m_i is the modal mass of the shell.

First, Fig. 3(a) shows the nonlinear resonance curves obtained for increasing values of the lateral point load. As observed, the response remains almost linear for low excitation levels, with well-defined resonance peaks around the system's natural frequency. Now, when the load amplitude is increased, the shell displays typical softening behavior with a jump for large values of load amplitude.

Next, by extending the modal combination 15 with an additional companion mode, we evaluate how this extra degree of freedom alters the nonlinear dynamics. The new terms is $W_{(1,n,c)}\, h(t)$, are added to the original 15 expansion to form the enriched basis.

Consequently, Fig. 3(b) displays the resonance curves for progressively larger vibration amplitudes. When comparing the resonance curves of Fig. 3(a) and Fig. 3(b), it is possible to see that curves there is not significative changes on the nonlinear dynamic response of the shell.

To analyze the nonlinear dynamic response of the empty shell subjected to a harmonic base excitation $A_g(t)$, Fig. 4. The result were obtained using the modal combination 15 plus companion mode, previously identified as exhibiting softening-type nonlinear behavior.

Table 2. Expansões modais para o campo de deslocamentos w, u e v

Expansão	**Modal Expansion**
15	$w = W_{(1,n)}\,h(t) + W_{(1,2n)}\,h(t) + W_{(2,2n)}\,h(t) + W_{(1,0)}\,h(t) + W_{(2,0)}\,h(t) + W_{(3,0)}\,h(t)$
	$u = U_{(1,n)}\,h(t) + U_{(1,2n)}\,h(t) + U_{(2,2n)}\,h(t) + U_{(1,0)}\,h(t) + U_{(2,0)}\,h(t) + U_{(3,0)}\,h(t)$
	$v = V_{(1,n)}\,h(t) + V_{(1,2n)}\,h(t) + V_{(1,3n)}\,h(t)$
17	$w = W_{(1,n)}\,h(t) + W_{(1,2n)}\,h(t) + W_{(2,2n)}\,h(t) + W_{(1,0)}\,h(t) + W_{(2,0)}\,h(t) + W_{(3,0)}\,h(t)$
	$+ W_{(4,0)}\,h(t)$
	$u = U_{(1,n)}\,h(t) + U_{(1,2n)}\,h(t) + U_{(2,2n)}\,h(t) + U_{(1,0)}\,h(t) + U_{(2,0)}\,h(t) + U_{(3,0)}\,h(t)$
	$+ U_{(4,0)}\,h(t)$
	$v = V_{(1,n)}\,h(t) + V_{(1,2n)}\,h(t) + V_{(1,3n)}\,h(t)$
18	$w = W_{(1,n)}\,h(t) + W_{(1,2n)}\,h(t) + W_{(2,n)}\,h(t) + W_{(2,2n)}\,h(t) + W_{(1,0)}\,h(t) + W_{(2,0)}\,h(t)$
	$+ W_{(3,0)}\,h(t)$
	$u = U_{(1,n)}\,h(t) + U_{(1,2n)}\,h(t) + U_{(2,n)}\,h(t) + U_{(2,2n)}\,h(t) + U_{(1,0)}\,h(t) + U_{(2,0)}\,h(t)$
	$+ U_{(3,0)}\,h(t)$
	$v = V_{(1,n)}\,h(t) + V_{(1,2n)}\,h(t) + V_{(2,n)}\,h(t) + V_{(2,2n)}\,h(t)$
19	$w = W_{(1,n)}\,h(t) + W_{(1,2n)}\,h(t) + W_{(2,2n)}\,h(t) + W_{(1,0)}\,h(t) + W_{(2,0)}\,h(t) + W_{(3,0)}\,h(t)$
	$+ W_{(4,0)}\,h(t) + W_{(5,0)}\,h(t)$
	$u = U_{(1,n)}\,h(t) + U_{(1,2n)}\,h(t) + U_{(2,2n)}\,h(t) + U_{(1,0)}\,h(t) + U_{(2,0)}\,h(t) + U_{(3,0)}\,h(t)$
	$+ U_{(4,0)}\,h(t) + U_{(5,0)}\,h(t)$
	$v = V_{(1,n)}\,h(t) + V_{(1,2n)}\,h(t) + V_{(1,3n)}\,h(t)$
43	$w = W_{(1,n)}\,h(t) + W_{(1,2n)}\,h(t) + W_{(1,3n)}\,h(t) + W_{(2,n)}\,h(t) + W_{(2,2n)}\,h(t) + W_{(3,n)}\,h(t)$
	$+ W_{(3,2n)}\,h(t) + W_{(1,0)}\,h(t) + \cdots + W_{(11,0)}\,h(t)$
	$u = U_{(1,n)}\,h(t) + U_{(1,2n)}\,h(t) + U_{(1,3n)}\,h(t) + U_{(2,n)}\,h(t) + U_{(2,2n)}\,h(t) + U_{(3,n)}\,h(t)$
	$+ U_{(3,2n)}\,h(t) + U_{(1,0)}\,h(t) + \cdots + U_{(11,0)}\,h(t)$
	$v = V_{(1,n)}\,h(t) + V_{(1,2n)}\,h(t) + V_{(1,3n)}\,h(t) + V_{(2,n)}\,h(t) + V_{(2,2n)}\,h(t) + V_{(3,n)}\,h(t)$
	$+ V_{(3,0)}\,h(t)$

All curves were obtained varying the amplitude y_0 ranging from 0.002 to 0.020 m. First, in Fig. 4(a), the resonance curve of the empty shell is displayed, and as can be seen, for small values of base displacement, the shell shows almost linear behavior but, if the base amplitude is increased, the resonance curves shows softening behavior with a jump at larger base displacement value. Under this condition, the resonance curve already displays nonlinear effects, including a shift of the peak toward lower frequencies, which is characteristic of softening behavior. These results indicate that, even without internal fluid, the structure exhibits significant nonlinearities under stronger base excitations, mainly due to modal coupling and geometric interactions captured by the selected modal combination.

Finally, Fig. 4(b) presents the nonlinear resonance curve of shell,now considering the presence of internal fluid with height $H_i = 0.25\,L$. The fluid has a density of $\rho_f = 1000\,\text{kg/m}^3$. The analysis was carried out using the same values of base displacement amplitude as in the previous case. Compared to the empty shell configuration, it can be clearly observed that the presence of fluid causes a shifting to the left the system's natural frequency. The linear analysis shows that the fundamental mode has $n = 7$ circumferential waves and its natural frequency is ω_0 = 179.734 rad/s.

This frequency shift indicates a significant influence of the fluid-structure interaction on the dynamic behavior of the shell. While the qualitative behavior of the resonance curve still exhibits softening characteristics, the

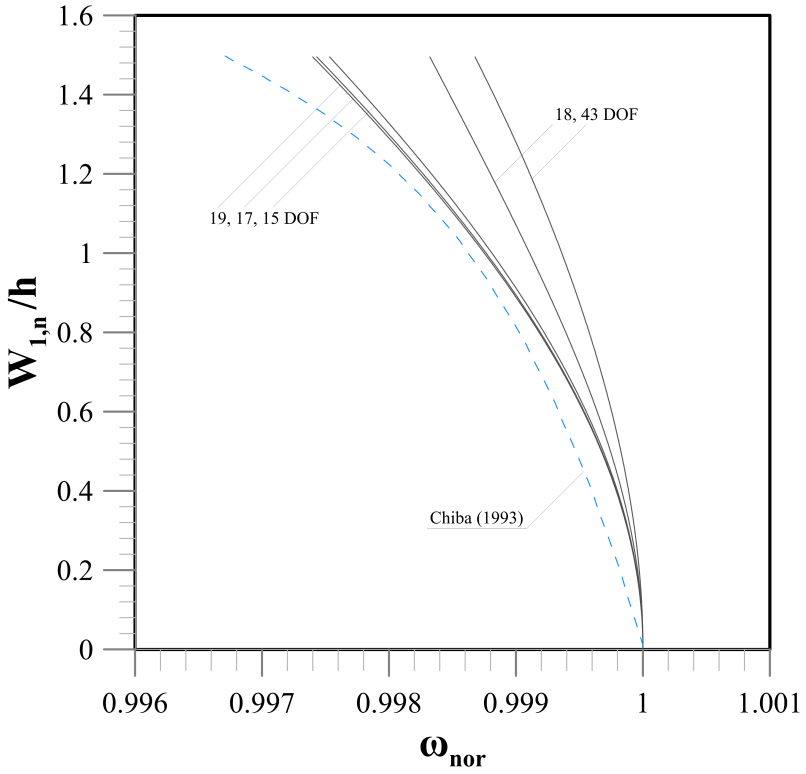


Figure 2. Frequency-amplitude relations of empty clamped cylindrical shell for selected field displacement expansions.

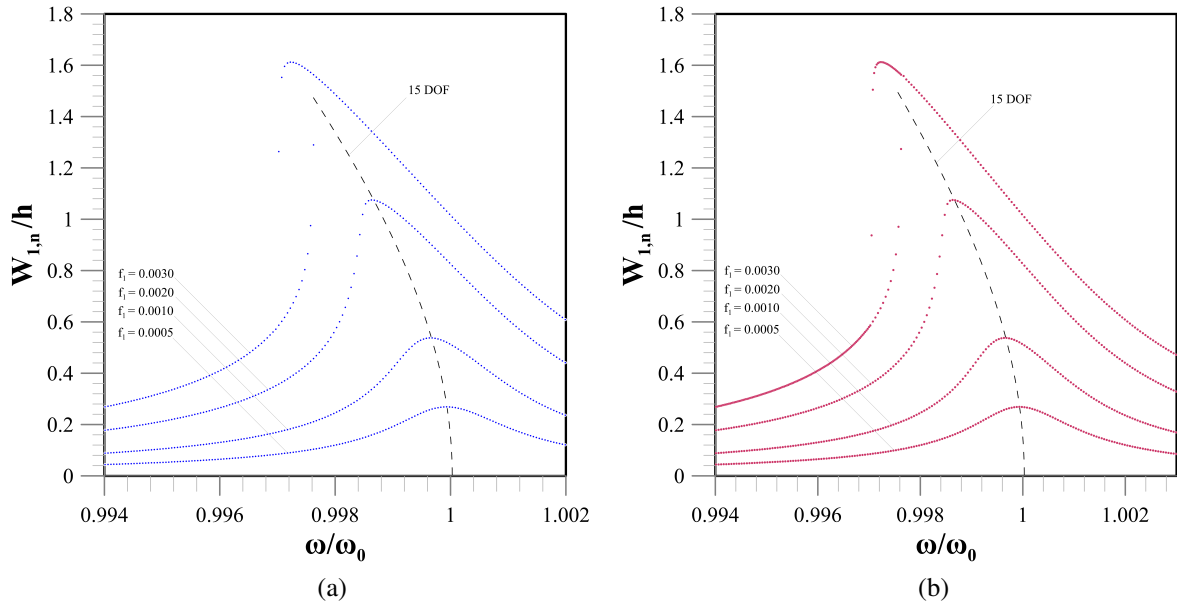


Figure 3. Nonlinear resonance curves of empty clamped cylindrical shell for lateral harmoinic load

resonance peak is displaced, highlighting the reduction of the fundamental natural frequency due to the added inertia and coupling introduced by the fluid. These results reinforce the importance of accounting for internal fluid effects when evaluating the nonlinear dynamic response of partially filled cylindrical shells.

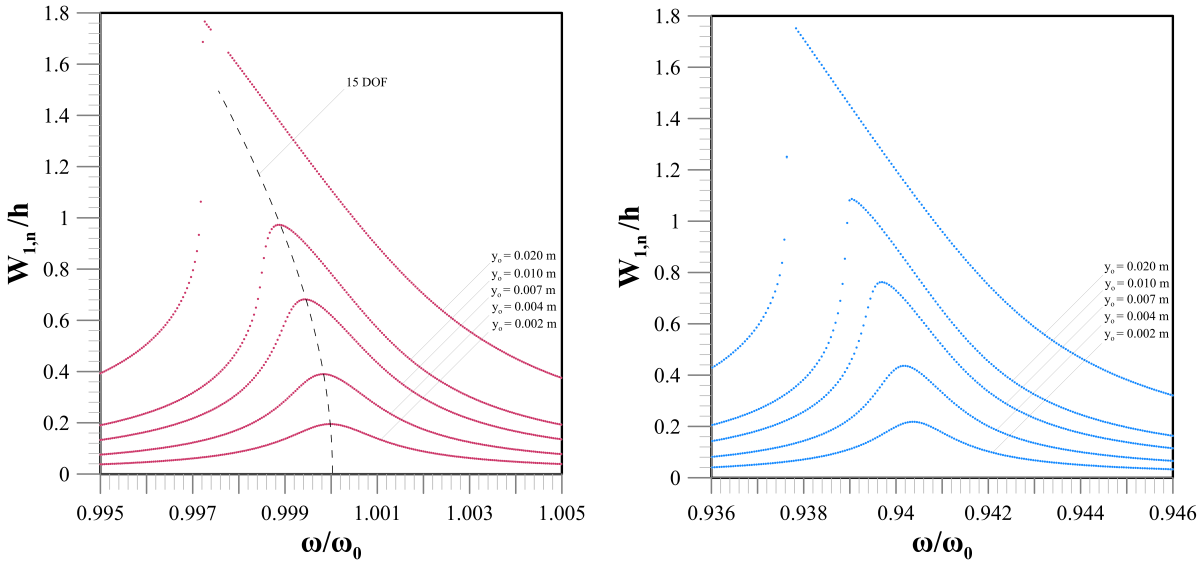


Figure 4. Nonlinear resonance curves for increasing base amplitude displacement. a) Empty shell and b) Partially fluid-filled shell.

4 Conclusions

In this work, the free and forced nonlinear vibrations of emptuy and partially fluid-filled clamped-free cylindrical shell are investigated under two independent excitations: a lateral point harmonic load and a harmonic base displacement. The Koiter–Sanders nonlinear shell theory is applied to obtain the strain energy, and the displacement fields are described using a double trigonometric series in both longitudinal and circumferential directions, satisfying the natural boundary conditions.

A convergence study was performed based on five selected modal combinations to identify the set of displacement fields capable of reproducing softening behavior in the frequency–amplitude curves. As a result, a configuration with 15 degrees of freedom was chosen due to its computational efficiency and its ability to generate nonlinear resonance curves in good agreement with experimental results presented by Chiba [4, 5]. In order to account for fluid–structure interaction effects, an additional accompanying mode was introduced.

The resonance curves were obtained for increasing values of each excitation applied separately. When the internal fluid occupies 25% of the shell's total length, the nonlinear dynamic response exhibits a clear shiffting in the natural frequencies of the system, indicating the influence of the added mass and inertia introduced by the fluid. The results highlight the importance of accounting for the presence of internal fluid in the accurate prediction of the nonlinear dynamic behavior of cylindrical shells.

References

[1] G. Warburton and J. Higgs. Natural frequencies of thin cantilever cylindrical shells. *Journal of Sound Vibration*, vol. 11, n. 3, pp. 335–338, 1970.

[2] C. Sharma. Free vibrations of clamped-free circular cylinders. *Thin-Walled Structures*, vol. 2, n. 2, pp. 175–193, 1984.

[3] H. Tottenham and K. Shimizu. Analysis of the free vibration of cantilever cylindrical thin elastic shells by the matrix progression method. *International Journal of Mechanical Sciences*, vol. 14, n. 5, pp. 293–310, 1972.

[4] M. Chiba. Non-linear hydroelastic vibration of a cantilever cylindrical tank—i. experiment (empty case). *International journal of non-linear mechanics*, vol. 28, n. 5, pp. 591–599, 1993a.

[5] M. Chiba. Non-linear hydroelastic vibration of a cantilever cylindrical tank—ii. experiment (liquid-filled case). *International journal of non-linear mechanics*, vol. 28, n. 5, pp. 601–612, 1993b.

[6] Y. Kurylov and M. Amabili. Nonlinear vibrations of clamped-free circular cylindrical shells. *Journal of sound and vibration*, vol. 330, n. 22, pp. 5363–5381, 2011.

[7] R. O. P. Montes. *Estudo Analítico-Numérico Para Análise Dinâmica Não Linear Acoplada Cascas Cilíndricas-Fluido*. PhD thesis, Universidade de Brasília, 2024.