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Combining ARX Models, Latent Force Representation, and Kalman Filter for Nonlinear Vibration Analysis

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Abstract. Identifying nonlinear mechanical systems plays a key role in structural dynamics, with applications in design, optimization, and monitoring systems in operation. In Structural Health Monitoring (SHM), identifying nonlinearities enables the detection, quantification, and characterization of damage. Although many methods have been developed, new techniques continue to emerge to simplify procedures and broaden applicability in structural engineering. Among these, combining physical modeling with machine learning has gained attention due to its flexibility, accuracy, and interpretability. This work proposes a novel methodology for characterizing nonlinearities in structural dynamic systems. The approach integrates a classical AutoRegressive model with eXogenous inputs (ARX) and the Kalman filter to extract nonlinear contributions from vibration signals. These contributions are modeled in the state-space equations using a Latent Force framework based on Gaussian Process models. The extracted nonlinear component can support damage detection and quantification, and it can be represented by a machine learning model to predict system behavior under varying operating conditions, including different vibration levels. The methodology is demonstrated on the classical Duffing oscillator, showing promising results in extracting the nonlinear restoring force and predicting system responses under previously unseen scenarios.

Keywords: Nonlinear System Identification, ARX Models, Latent Force Modeling, Kalman Filter.

1 Introduction

Many system identification methods rely on assumptions of linearity. Although linear models provide simple and analytical solutions, the behavior of many mechanical systems, especially those operating under extreme conditions or experiencing some level of degradation (e.g., due to damage), is intrinsically nonlinear. In such cases, nonlinear system identification techniques are required to ensure accuracy and reliability in estimating the nonlinear contributions and in predicting the system's behavior under previously unseen conditions [1]. Several methodologies have been developed in this field. Some rely on classical approaches, such as NARX, NARMAX, and Volterra series-based models. Others employ machine learning and deep learning techniques, including GP-NARX and LSTM, among others. Recently, hybrid methodologies that combine classical methods with machine learning models have gained significant attention [2].

Gaussian Processes are a widely used machine learning approach, particularly suitable for time series prediction. In the context of mechanical oscillatory systems, certain system states, such as input forces or nonlinear restoring forces, can be modeled as Gaussian Processes. This type of representation is known as latent force modeling. Fortunately, a Gaussian Process in time, with zero mean and a stationary covariance function, can be represented by a stochastic differential equation, which can, in turn, be expressed as a linear Gaussian state-space model [3]. This enables the use of the latent force representation to model nonlinear components in conjunction with a classical linear model (e.g., an ARX model), resulting in an augmented state-space formulation.

Therefore, this study presents a methodology for identifying nonlinearities in structural dynamic systems by combining an ARX model with the Kalman filter. Nonlinear contributions are modeled in the state-space equations using a latent force approach based on Gaussian Processes. The method enables damage characterization and prediction of system behavior under varying conditions. Its effectiveness is demonstrated on a Duffing oscillator.

2 Nonlinear Dynamic Modelling

The ARX model is widely used to identify linear dynamic systems. The formulation assumes that the response of a system at a given discrete time instant t_k can be described as a linear relationship between past observations of the system's output and input signals [4]. Considering the state vector $\mathbf{x}(t_k) = [y(t_{k-1}),\ y(t_{k-2}),\ \ldots,\ y(t_{k-n_y})]^{\mathrm{T}}$, and the input vector $\mathbf{u}(t_k) = [u(t_k),\ u(t_{k-1}),\ \ldots,\ u(t_{k-n_u+1})]^{\mathrm{T}}$, the ARX model can be written in state-space form as

$$\mathbf{x}(t_{k+1}) = \mathbf{A}\,\mathbf{x}(t_k) + \mathbf{B}\,\mathbf{u}(t_k), \tag{1}$$

$$y(t_k) = \mathbf{C}\,\mathbf{x}(t_k) + \mathbf{D}\,\mathbf{u}(t_k), \tag{2}$$

with the matrices having the form

$$\mathbf{A} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n_y-1} & -a_{n_y} \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},\ \mathbf{B} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_u} \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix},\ \mathbf{D} = 0\,, \tag{3}$$

where $y(t_k)$ and $u(t_k)$ are, respectively, the output and input of the linear system at time t_k, n_y is the number of past output delays considered, and n_u is the number of past input delays. The terms a's and b's represent the model coefficients. This model provides excellent predictive performance for the behavior of dynamic systems, as long as the linearity assumption holds. The coefficients can be previously estimated using ridge linear regression.

Considering that the nonlinear restoring force $f_{nl}(t)$ of a dynamic system can be modeled as a Gaussian Process with zero mean and a stationary covariance function, this component can be represented by a linear time-invariant stochastic differential equation. Considering a Matérn 3/2 covariance function and rearranging the corresponding equation into state-space form, we obtain [5]:

$$\begin{bmatrix} \dot{f}_{nl}(t) \\ \ddot{f}_{nl}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & -2\lambda \end{bmatrix} \begin{bmatrix} f_{nl}(t) \\ \dot{f}_{nl}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t), \tag{4}$$

where $w(t)$ is a white noise process and $\lambda = \frac{\sqrt{3}}{l}$, with l being the covariance function lengthscale. The white noise process has spectral density defined as $q = \frac{2\sigma_f^2 \pi^{\frac{1}{2}} \Gamma(2) \lambda^3}{\Gamma\left(\frac{3}{2}\right)}$, with σ_f^2 being the covariance function variance, and $\Gamma(.)$ representing the gamma function.

By discretizing the model presented in eq. (4) and combining it with the representation given in eqs. (1) and (2), an augmented state-space model can be constructed. This formulation allows the estimation of the hidden states using the Kalman filter [6]. To enable this, the coefficients of eq. (3) must be estimated in advance. The hyperparameters of the Gaussian Process representation can be inferred jointly with the hidden states using a Bayesian inference method. Further details can be found in Rogers and Friis [3], where a similar methodology is employed, but using a physical model to represent the system's linear components. The use of an underlying ARX model can offer advantages when dealing with systems with multiple degrees of freedom and noisy data, while also providing a certain level of physical interpretability.

3 Identification of a Duffing Oscillator

All the simulations were carried out considering a Duffing oscillator

$$m\ddot{x}(t) + c\dot{x}(t) + k_1 x(t) + k_2 x(t)^2 + k_3 x(t)^3 = U(t), \tag{5}$$

with the following parameters: $m = 0.26$ [kg], $c = 1.36$ [Ns/m], $k_1 = 5490$ [N/m], $k_2 = 32400$ [N/m^2] and $k_3 = 46800000$ [N/m^3] [7]. The ARX model was identified using $n_y = 8$ and $n_u = 6$, with the system operating in a linear regime (i.e., under low-amplitude excitation). Figure 1 shows the performance of the linear model compared to the actual system response under a high-amplitude excitation, where nonlinear behavior is evident. It is clear that the linear model fails to accurately predict the system output under these conditions.

Then, to filter the nonlinear contribution from the system response and estimate the nonlinear restoring force, the methodology based on state-space representation, latent force modeling, and the Kalman filter was applied.

First, the hyperparameters of the Matérn kernel and the signal noise variance were estimated using a Bayesian approach, specifically employing the standard Metropolis random walk MCMC algorithm [8]. Using the estimated hyperparameters, the nonlinear restoring force was inferred through the Kalman filter algorithm.

Figures 2 and 3 show, respectively, the filtered output signal obtained using the proposed algorithm and the estimated nonlinear restoring force, both compared to the corresponding true signals. As can be observed, the estimates are very close to the true responses. It is worth noting that this is a simulated study, and additional tests using experimental data are currently being conducted to further validate the proposed approach.

The estimation of the true nonlinear restoring force is, in itself, an interesting result, as it can often be used to monitor the presence or progression of damage in nonlinear systems [9]. However, in this work, we take a step further. By exploring the relationship between the delayed output signals and the nonlinear restoring force, we train a Gaussian Process Regression model to predict the system response for previously unseen inputs. Figure 4 shows the prediction obtained from the estimated model compared to the true response, considering a high-level random input, i.e., a signal with characteristics different from those used during the filtering step. Additionally, the Gaussian Process provides confidence bands for the prediction. The agreement between the predicted and true responses demonstrates the generalization capability of the proposed model.

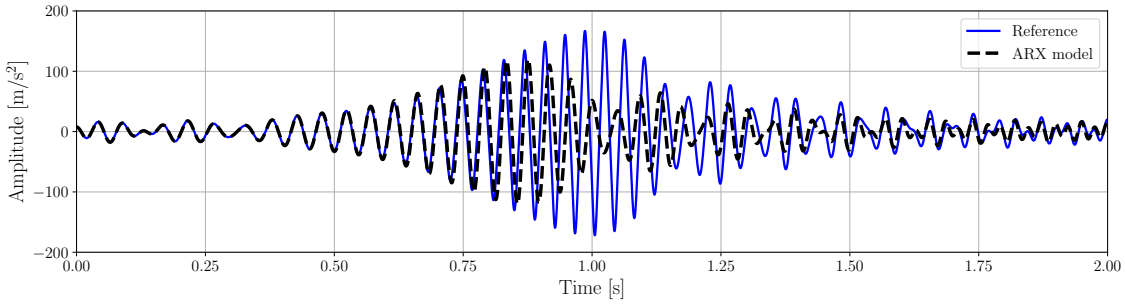


Figure 1. ARX model output under high-amplitude excitation.

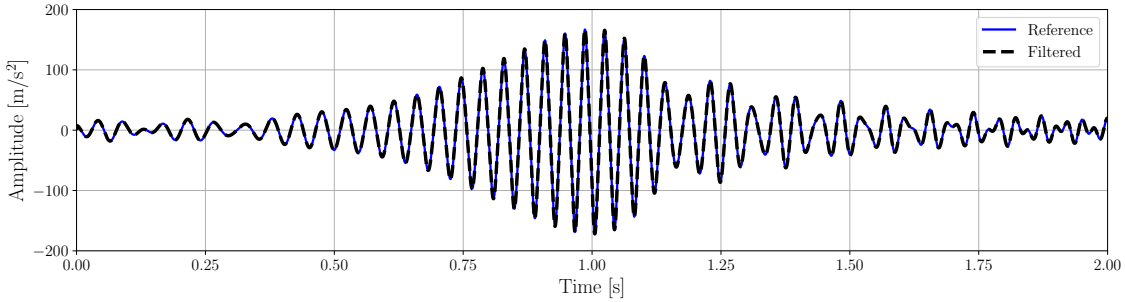


Figure 2. Filtered output signal.

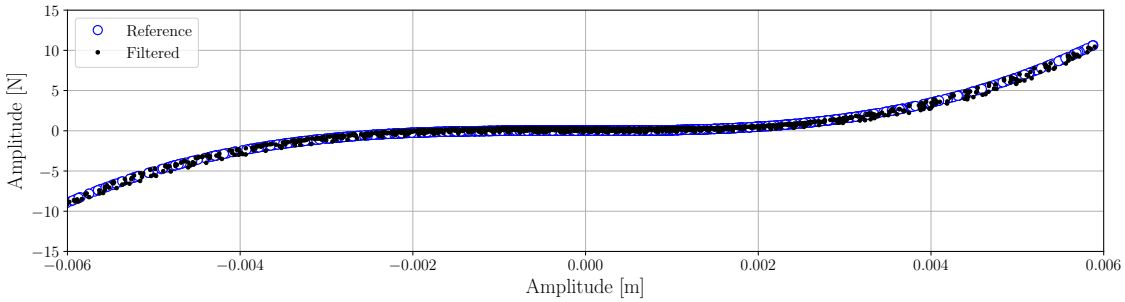


Figure 3. Estimated nonlinear restoring force.

4 Conclusions

This work explored the use of the state-space representation of latent forces in a new context. Instead of employing the representation typically associated with a continuous-time oscillator model, it was combined with

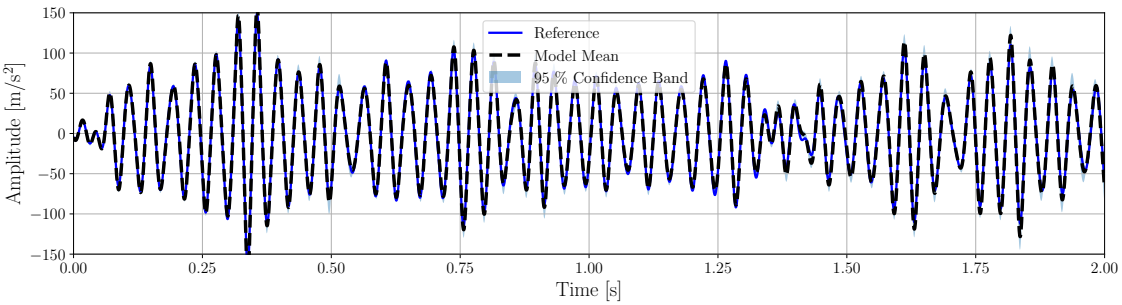


Figure 4. Validation of the model identified.

an ARX model to filter the nonlinear components present in the dynamic response of a Duffing oscillator. By explicitly modeling the nonlinear component, this hybrid strategy provides opportunities for advanced tasks such as damage detection, quantification, and the prediction of system responses across a range of operating conditions, even at unseen vibration levels. The promising results obtained from simulations highlight its potential as a flexible and computationally efficient methodology for modeling and filtering nonlinear dynamic behavior. Nevertheless, further validation with experimental data is necessary to confirm its robustness in practical applications. **Acknowledgements.** The first author is grateful for the support of Universidade Federal do Espírito Santo (UFES). The authors also would like to thank the support of the INCT-EIE (National Institute of Science and Technology - Smart Structures in Engineering), funded by the Brazilian agencies CNPq under grant number 406148/2022-8, as well as CAPES (Coordination for the Improvement of Higher Education Personnel) and FAPEMIG (Minas Gerais State Research Support Foundation).

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