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Reliability-based multi-material topology optimization with cubic spline interpolation

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Abstract. Topology Optimization (TO) aims to enhance structural performance by determining the optimal material distribution within a fixed design domain, thereby maximizing efficiency and minimizing cost. This paper investigates the application of the Smoothing Evolutionary Structural Optimization (SESO) method to multi-material topology optimization problems, integrating it with the Reliability-Based Topology Optimization (RBTO) framework. The primary contribution of this study lies in the extension of the SESO method to multi-material settings through the implementation of cubic spline interpolation for estimating both Young's moduli E and cost functions C as functions of material density ρ. The proposed algorithm, developed in MATLAB, also computes the first and second derivatives of the interpolated functions and applies a simple Gaussian filter to smooth the output fields. This promotes a more continuous material transition and improves numerical stability throughout the optimization process. Classical three-dimensional TO benchmark problems are evaluated, including a short cantilever beam and the MBB beam, considering steel, aluminum, and concrete as candidate materials.The numerical results demonstrate substantial reductions in structural cost and improvements in global mechanical performance, particularly under uncertainty in design parameters. The proposed technique proves to be a promising approach for practical applications involving the simultaneous selection and spatial distribution of multiple materials in optimized structures. Ultimately, it contributes to the development of more sustainable and high-performance solutions in structural engineering.

Keywords: Multi-Material Topology Optimization, Reliability-Based Topology Optimization, Cubic Spline.

1 Introduction

Topology Optimization (TO) has emerged as a powerful computational framework in engineering design, particularly for structural applications, where it aims to achieve optimal material distribution within a fixed domain. Beyond maximizing structural efficiency, TO also seeks to minimize material usage and operational costs, making it an essential component in sustainable and performance-driven engineering, Zhang et al. [1].

A growing branch of TO is Multi-Material Topology Optimization (MMTO), which enables the allocation of different materials typically high-performance, high-cost materials in critical stress regions, and lower-cost alternatives in less demanding areas. This layered material strategy can lead to more efficient and cost-effective

designs but introduces additional challenges in both modeling and fabrication. Recent advances in additive manufacturing, such as point-wise and layer-by-layer printing, have made the practical realization of MMTO feasible, Bandyopadhyay and Heer [2]; Han and Lee [3].

Several methodologies have been developed to support these objectives. Among them are the Evolutionary Structural Optimization (ESO) method by Xie and Steven [4], the Solid Isotropic Material with Penalization (SIMP) approach by Bendsøe and Sigmund [5], and the more recent Smoothing Evolutionary Structural Optimization (SESO) technique by Simonetti et al. [6]. SESO enhances the classical ESO by introducing a bidirectional heuristic for the removal and reintroduction of elements, which differs fundamentally from BESO and contributes to better convergence and solution stability.

Despite these developments, many classical TO formulations remain limited by deterministic assumptions and single-material domains, failing to account for real-world uncertainties in material properties, loading conditions, and geometric parameters. To address these issues, Reliability-Based Topology Optimization (RBTO) was introduced, incorporating stochastic variables to model such uncertainties, Kharmanda et al. [7].

Parallel to the development of RBTO, MMTO has gained increasing attention. Recent innovations include level-set methods Sha et al. [8], multi-resolution optimization Jiang et al. [9], material cost integration Doan et al. [10], and stress-constrained formulations like ordered SIMP, Zuo and Saitou [11]. The field has also expanded to include thermal coupling effects Chen et al. [12] and functionally graded materials Li et al. [13].

Building upon these efforts, Simonetti et al. [14] explored 3D MMTO with sigmoid-based filtering to eliminate intermediate density artifacts, while Silva et al. [15] extended SESO for 2D MMTO under uncertainty by incorporating FORM-based reliability analysis demonstrating the potential of SESO for robust and manufacturable structural design.

The core contribution of the present study lies in the integration of cubic spline interpolation into the SESO-RBTO framework for modeling material properties such as Young's modulus (E) and cost (C) as smooth functions of material density (ρ). Unlike linear or penalized interpolations, the spline-based approach ensures smooth transitions between material phases, reduces artificial stress concentrations, and enhances numerical stability particularly beneficial in MMTO contexts involving multiple materials.

The developed MATLAB algorithm computes first and second derivatives of the interpolated fields and applies Gaussian filtering to the sensitivities, improving convergence and robustness of the optimization. Through classical 3D benchmark examples, we demonstrate that the proposed method significantly reduces structural compliance and cost while maintaining high reliability levels. This integrated strategy advances the capabilities of TO under uncertainty, providing a practical and efficient pathway for sustainable high-performance structural design.

2 Multi-material TO using Cubic spline interpolation

The interpolation of material properties as a function of the fictitious density ρ is a central aspect in density-based TO formulations, as it ensures a continuous transition between solid and void regions in structural design problems. While classical approaches such as the SIMP method adopt a penalized monomodal interpolation of the form $E(\rho) = \rho^p E_0$, this formulation presents notable limitations, including lack of smoothness in its derivatives, numerical instabilities near material boundaries, and the emergence of intermediate zones with physically unrealistic properties. As a promising alternative, this work adopts a cubic spline interpolation capable of representing the variation of the Young's modulus $E(\rho)$ and cost function $C(\rho)$ with greater smoothness and accuracy.

This approach is particularly advantageous in multi-material TO frameworks such as the SESO method, as it enables smooth transitions between distinct materials without introducing fictitious intermediate phases, and facilitates the computation of continuous derivatives required for sensitivity analysis. The density domain $[\rho_i, \rho_n]$ is partitioned into subintervals $[\rho_i, \rho_{i+1}]$, where a cubic polynomial is defined in each interval of the form:

$$E(\rho) = \sum_{k=0}^{3} a_{ik}(\rho - \rho_i)^k \, , to \; \rho \in [\rho_i, \rho_{i+1}] \tag{1}$$

with a_{ik} being the coefficients of the cubic polynomial associated with the interval $[\rho_i, \rho_{i+1}]$, with i denoting the index of the subinterval in which the density ρ lies, and n being the total number of interpolation points (or nodes) used to define the full interpolation range. Thus, to cover the entire density domain $[\rho_i, \rho_n]$, is expressed as:

$$E(\rho) = \sum_{i=1}^{n-1} \sum_{k=0}^{3} a_{ik} (\rho - \rho_i)^k x_i(\rho) \tag{2}$$

i denotes the index of the subinterval in which the density ρ lies, and n is the total number of interpolation points (or nodes) used to define the intervals. The function $x_i(\rho)$ is defined as:

$$x_i(\rho) = \begin{cases} 1 \; if \;\; \rho \in [\rho_i, \rho_{i+1}] \\ 0 \;\; otherwise \end{cases} \tag{3}$$

The coefficients a_{ik} are determined by enforcing continuity conditions on the value, first derivative, and second derivative of the interpolated function at each node of the interpolation mesh:

$$\begin{gathered} E(\rho_i) = E_i \;\; and \;\; E(\rho_{i+1}) = E_{i+1} \\ \frac{\partial E}{\rho_{i+1}^-} = \frac{\partial E}{\rho_{i+1}^+} \\ \frac{\partial^2 E}{(\rho_{i+1}^-)^2} = \frac{\partial^2 E}{(\rho_{i+1}^+)^2} \\ \frac{\partial^2 E}{(\rho_1)^2} = 0 \; and \; \frac{\partial^2 E}{(\rho_n)^2} = 0 \end{gathered} \tag{4}$$

where E is the leading term in eq. (5), representing the sensitivity of the optimization problem, and it must be replaced by:

$$\frac{\partial E}{\partial \rho} = \sum_{i=1}^{n-1} \sum_{k=0}^{3} k a_{ik} (\rho - \rho_i)^{k-1} x_i(\rho) \tag{5}$$

with the second derivative is expressed by:

$$\frac{\partial^2 E}{\partial \rho^2} = \sum_{i=1}^{n-1} \sum_{k=0}^{3} k(k-1) a_{ik} (\rho - \rho_i)^{k-2} x_i(\rho) \tag{6}$$

Cubic spline interpolation is also applied to the cost function C(ρ), analogously to the interpolation of the elasticity modulus. This technique ensures continuity and smoothness up to the second derivative, outperforming linear interpolation by avoiding abrupt transitions and sharp corners. However, it introduces a higher computational cost and requires validation of the density values to ensure interpolation occurs only within valid intervals. For two-material systems, a single smooth spline curve is sufficient to reduce stress concentrations and improve structural performance (i.e., lower compliance). In contrast, when three materials are involved, multiple interdependent spline curves are required, increasing both the complexity and the number of combinations, as well as the computational effort of the optimization process.

3 Reliability Analysis: First-Order Reliability Method (FORM)

According to [7], the design variables are treated as deterministic, since they correspond to controllable system parameters such as geometric dimensions, material properties, and loading conditions as well as to probabilistic model parameters, such as mean values and standard deviations of random variables. On the other hand, stochastic physical variables, which influence the failure mechanisms, may represent uncertainties related to geometry, material properties, or applied loads. These structural uncertainties are modeled through probabilistic distributions, allowing for a more realistic representation of the system's behavior under variability. The reliability-based

optimization problem is coupled with the SESO algorithm and solved using a hierarchical formulation with two nested subproblems embedded within a single topology optimization loop, given by:

$$\begin{aligned} &\textit{minimize} && C(x) = \frac{1}{2} f^T u \\ &\textit{subject to} && g_k(x) \leq 0 \quad \forall\, k = 1,2,\dots,K \\ & && \beta(x,u) \geq \quad \beta_t \end{aligned} \tag{7}$$

where C(x) denotes the objective function (compliance), $g_k(x) \leq 0$ represents the set of k constraint functions, $\beta(x,u)$ is the structural reliability index, and β_t is the target reliability level. The reliability analysis employs the reliability index $\beta(x,u)$, defined as the minimum distance from the limit-state function $H(x,u)$ to the origin in the standard normal space of the random variables. This index is obtained by solving a minimization problem that identifies the most probable point of failure on the limit-state surface. It is therefore a reduced-space optimization problem, whose solution provides a quantitative measure of the structural reliability, given by:

$$\begin{aligned} &\textit{minimize} && \beta(x,u) = d(u) = \sqrt{\sum_{i=1}^{n} (u_i)^2} \\ &\textit{subject to} && H(x,u) \leq 0 \end{aligned} \tag{8}$$

In this context, d(u) denotes the distance in the standard normal space of the random variables, and $H(x,u)$ is the performance (or limit-state) function defined in that space. The condition $H(x,u) < 0$ characterizes failure, whereas $H(x,u) = 0$ defines the boundary between the safe and failure domains, i.e., the limit-state surface (see Fig. 1b). In the physical space, the equivalent function is $G(x,y)$, which represents the mapped image of $H(x,u)$ through the transformation from normalized random variables to their corresponding physical representations (see Fig. 1a).

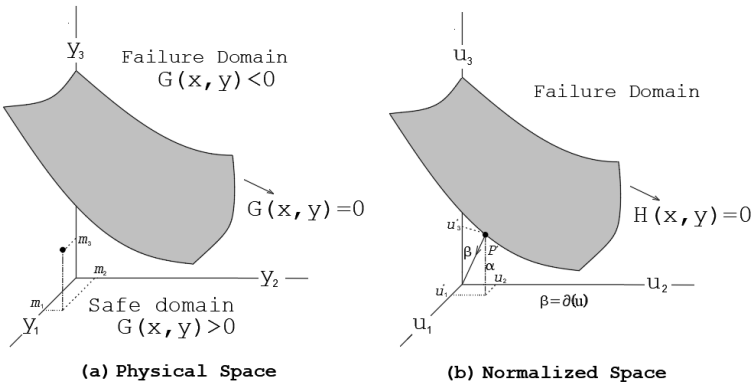


Figure 1. (a) Physical Space and (b) Normalized Space

4 Numerical Examples

The proposed formulation is validated through two benchmark structural TO problems. To evaluate the performance of the SESO-Spline-DTO algorithm, using also the deterministic version, simulations were conducted using two materials: steel ($E = 200$ GPa, ρ = 7860 kg/m³) and concrete ($E = 50\,GPa$, ρ = 2400 kg/m³), both with Poisson's ratio of $\nu = 0.30$. The finite element models employed linear hexahedral elements, and all computations were performed on a Core i7-2370, 8th Gen notebook (2.8 GHz, 20 GB RAM). This method does not require predefined material zones; instead, the material distribution emerges naturally through cubic spline interpolation as a function of the local pseudo-density ρ. Elements with ρ close to 1 acquire steel-like properties (highlighted in red), while lower values correspond to concrete-like behavior (depicted in blue). This strategy enables continuous transitions between materials and supports adaptive, performance-driven multi-material configurations. In both examples, the random variables considered were volume, Young's modulus, loading, and geometry each modeled as a normally distributed variable with a standard deviation of 10% of the mean value.

4.1 Cantilever Beam

The example considers a cantilever beam subjected to a vertical load of $P = 1{,}000\ kN$, distributed along half of the right edge. The structural domain with dimensions $L_x = 64\ dm$, $L_y = 40\ dm$ e $L_z = 2\ dm$ is discretized using 3,200 finite elements. Figures 2b and 2c show the optimal configurations using the SESO-Spline-DTO model and the SESO-Spline-RBTO formulation for MMTO. In the deterministic analysis, the final volume was 30% of the initial volume with a total strain energy of $254.9\ kJ$. In contrast, the SESO-Spline-RBTO formulation achieved a reduced volume of 25% while maintaining an energy of $255.3\ kJ$. Notably, the RBTO procedure using FORM converged in only two iterations, demonstrating its robustness and efficiency in obtaining lightweight topologies with enhanced performance. The computational cost was 496 seconds for the RBTO analysis, compared to 440 seconds for the deterministic case.

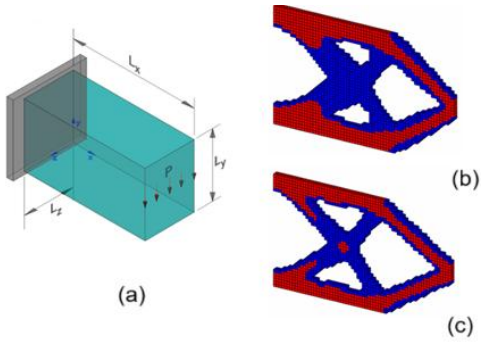


Figure 2. (a) Initial Domain (b) optimal topology: MMTO-SESO-Spline-DTO; (c) optimal topology: MMTO-SESO-Spline-RBTO

4.2 Simple suport beam

The 3D MBB beam is evaluated herein, with the design domain and boundary conditions are shown in Fig. 3a, with dimensions $Lx = 120dm$, $Ly = 20\ dm$ e $Lz = 2\ dm$. The domain was discretized into 4,800 finite elements, following the configuration adopted by Liu et al. [16], and a point load of 100 kN was applied at the center of the top surface. The limit state function adopted was that proposed by [7]. The SESO-Spline-DTO and SESO-Spline-RBTO formulations to MMTO demonstrated robustness, yielding optimal topologies with material savings. Fig. 3b shows the optimal topology obtained using the SESO-based DTO model, while Fig. 3c presents the optimal 3D MBB beam topology for a target reliability index of $\beta t = 3$, corresponding to a failure probability of 0.13%.

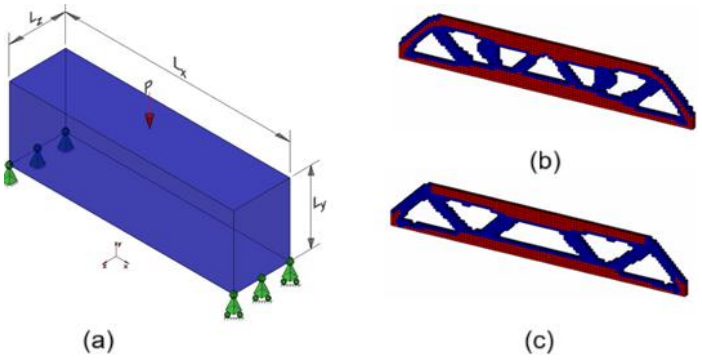
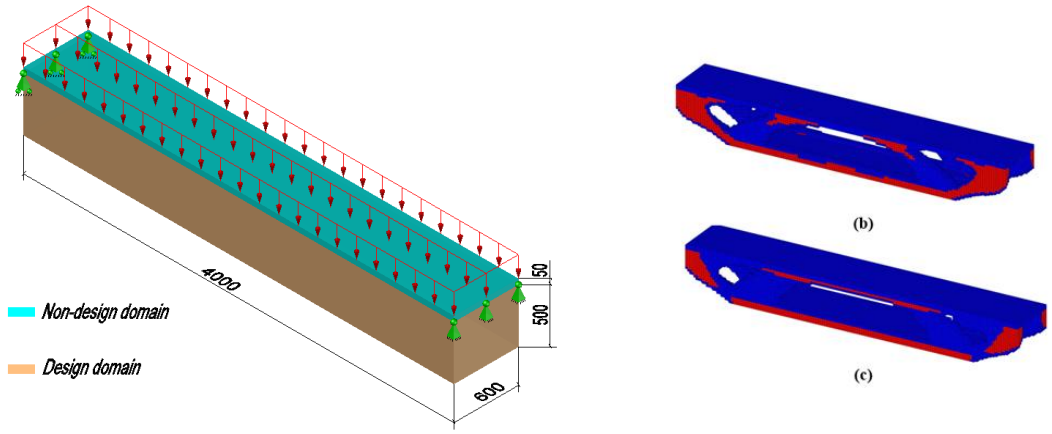


Figure 3. (a) Initial Domain (b) optimal topology: MMTO-SESO-Spline-DTO (c) optimal topology: MMTO-SESO-Spline-RBTO

4.3 Bridge Structure

In a steel-concrete composite bridge structure, steel is typically used in the tension regions and concrete in the compression regions, maximizing the mechanical properties of both materials. This approach results in a robust and efficient solution, since the materials can be fabricated in sections, as proposed by Li et al. [17]. In this context, the design domain, see Fig. 4a, of a 4-m span bridge was topologically optimized using the SESO-Spline and SESO-Linear methodologies. The evaluated bridge has dimensions of 4 m × 0.6 m × 0.55 m, with a deck thickness of 0.05 m, as shown in Fig. 4b and 4c, respectively.



(a)

Figure 2. (a) Initial Domain (b) optimal topology: MMTO-SESO-DTO; (c) optimal topology: MMTO-SESO-Spline-RBTO

The results in Table 1 show that cubic spline interpolation achieved significantly higher convergence rates compared to linear interpolation, in addition to exhibiting a more stable evolution. Although the computational cost increased due to the larger number of iterations, the gains in robustness justify the use of cubic spline interpolation.

Table 1. Comparison table: SESO-Linear versus SESO-Spline for bridge problem

Methods	**SESO-Linear DTO**	**SESO-Linear RBTO Normal**	**SESO-Spline DTO**	**SESO-Spline RBTO Normal**
Computational cost (s)	8769.6	9320.2	17998.9	18065.7
Convergence rate	0.001	0.001	0.298	0.199
N° Iteration at Convergence	104	109	199	200
Compliance (Nm)	0.27	0.26	0.34	0.36
Cost	0.11	0.09	0.11	0.08

5 Conclusions

This study successfully extended the SESO methodology to multi-material topology optimization (MMTO) by integrating cubic spline interpolation within the RBTO framework. The proposed interpolation scheme enabled

smooth and continuous transitions of both the elastic modulus and cost functions with respect to material density, which contributed to enhanced numerical stability and improved structural performance. The MATLAB-based implementation proved robust in handling multiple material distributions, as evidenced by the presented numerical examples.

The results demonstrated significant cost reductions and improved structural efficiency, particularly under design parameter uncertainty, confirming the effectiveness of the spline-based formulation. While cubic spline interpolation outperformed linear methods in terms of accuracy and robustness, it also required a higher computational cost due to the increased number of materials and iterations. Future research may focus on adaptive interpolation strategies and parallel computing to alleviate computational demands. Overall, the proposed method shows strong potential for practical applications, paving the way toward more sustainable and high-performance structural designs.

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References

[1[Zhang, W., J. Zhu, and T. Gao, Topology Optimization in Engineering Structure Design, 2016.
[2] Bandyopadhyay, A.; B. Heer, Additive manufacturing of multi-material structures, Mater. Sci. Eng. R 129 (2018) 1–16.
[3] D. Han, H. Lee, Recent advances in multi-material additive manufacturing: Methods and applications, Current Opin. Chem. Eng. 28 (2020) 158–166.
[4] Xie, Y. M., & Steven, G. P. (1993). A simple evolutionary procedure for structural optimization. Computers & structures, 49(5), 885-896.
[5] Bendsøe, M. P. and Sigmund, O., Material interpolation schemes in topology optimization, Archive of Applied Mechanics, Vol.69, (1999), pp.635–654.
[6] Simonetti, H. L., Almeida, V. S., & de Oliveira Neto, L. (2014). A smooth evolutionary structural optimization procedure applied to plane stress problem. Engineering structures, 75, 248-258.
[7] Kharmanda, G.; Olhoff, N.; El-Hami, A. (2004): Optimum safety factor approach for reliability- based design optimization with extension to multiple limit state case, Structural and Multidisciplinary Optimization, Vol 27,421-434.
[8] Sha, W., Xiao, M., Gao, L., & Zhang, Y. (2021). A new level set based multi-material topology optimization method using alternating active-phase algorithm. Computer Methods in Applied Mechanics and Engineering, 377, 113674.
[9] Jiang, Y., Zhan, K., Xia, J., & Zhao, M. (2023). Topology optimization for minimum compliance with material volume and buckling constraints under design-dependent loads. Applied Sciences, 13(1), 646.
[10] Doan, Q. H., Lee, D., Lee, J., & Kang, J. (2020). Multi-material structural topology optimization with decision making of stiffness design criteria. Advanced Engineering Informatics, 45, 101098.
[11] Zuo, W., & Saitou, K. (2017). Multi-material topology optimization using ordered SIMP interpolation. Structural and Multidisciplinary Optimization, 55(2), 477-491.
[12] Chen, J., Zhao, Q., & Zhang, L. (2022). Multi-material topology optimization of thermo-elastic structures with stress constraint. Mathematics, 10(8), 1216.]
[13] Li, J., Gao, L., Ye, M., Li, H., & Li, L. (2022). Topology optimization of irregular flow domain by parametric level set method in unstructured mesh. Journal of Computational Design and Engineering, 9(1), 100-113.
[14] Simonetti, H. L., Neves, F. d. A. d., Almeida, V. S., Silva, M. M. d., & Neto, L. d. O. (2025). Three-Dimensional Multi-Material Topology Optimization: Applying a New Mapping-Based Projection Function. Materials, 18(5), 997. https://doi.org/10.3390/ma18050997.
[15] da Silva, M. M., das Neves, F. D. A., Simonetti, H. L., & da Rocha Freitas, M. S. (2024, December). Multi-material topology optimization of 2D structures using the SESO and SIMP method with reliability. In Ibero-Latin American Congress on Computational Methods in Engineering (CILAMCE).
[16] Liu, K., & Tovar, A. (2014). An efficient 3D topology optimization code written in Matlab. Structural and multidisciplinary optimization, 50(6), 1175-1196.
[17] Li Y., Wu H., Xie X., Zhang L., Yuan P. F., Xie Y. M., "FloatArch: A cable-supported, unreinforced, and re-assemblable 3D-printed concrete structure designed using multi-material topology optimization."*Additive Manufacturing*, Volume 81, 2024.