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Efficient Time-Dependent Analysis of Reinforced and Prestressed Concrete Using Rate-Type Formulations

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Abstract. This study examines the time-dependent behavior of reinforced and prestressed concrete structures using rate-type formulations. The integral-type approach requires storing the full load history and reprocessing all past stress increments at each time step, resulting in high computational cost. This cost can be substantially reduced by adopting a rate-type formulation, which approximates the creep function with a Dirichlet series derived from a Kelvin chain approximation. These series are capable of accurately reproducing creep functions obtained from laboratory tests or prescribed by design codes. The stress–strain rate-type law for aging concrete is expressed by a second-order differential equation, but a single rate-type incremental equation is derived from a Kelvin chain approximation, considering the superposition principle. The paper details the numerical procedure used to integrate the rate-type equations over time, using a fixed time discretization. A relaxation model for prestressing steel is also adopted, accounting for the influence of varying strain histories on stress relaxation, an aspect often neglected in traditional approaches. The proposed formulation improves numerical efficiency while maintaining accuracy in long-term structural analysis, as verified through a representative example.

Keywords: rate-type formulation, concrete creep, Kelvin chain approximation, prestressing steel relaxation.

1 Introduction

The properties of concrete evolve over time, but its time-dependent behavior—particularly creep—is generally assumed to be approximately linear within the service stress range. Extending Boltzmann's superposition principle to aging materials leads to an integral-type creep law for variable stresses. Although this formulation is accurate, it requires storing and processing the entire stress history at each time step, resulting in high computational cost.

This limitation can be overcome by adopting a rate-type formulation, in which the creep compliance function is approximated by a Dirichlet series. The stress–strain relationship is then expressed in rate form, allowing for incremental computations that are both efficient and scalable.

This presentation provides a simplified overview of a research project on the time-dependent analysis of reinforced and prestressed concrete. A complete discussion of both approaches, including detailed derivations of an efficient rate-type formulation, representative examples, and implementation strategies, is available in Reference [1].

Bažant and Jirásek [2, 3] proposed a rate-type model for aging concrete using discrete Kelvin chains. Their formulation describes the behavior of an aging Kelvin unit through a second-order differential equation, assuming constant stiffness moduli at the midpoint of each time interval.

Building on this foundation, the present study derives a simplified incremental equation based on the rate-type creep law and the superposition principle, avoiding higher-order differential terms. Compliance values are interpolated linearly within each interval to improve accuracy while maintaining compatibility with standard finite

element procedures. This approach facilitates the upgrade of existing programs from integral-based to rate-based formulations.

Stress relaxation under varying strain in prestressing steel is modeled using the method proposed by Glodowski, Hoff, and Fox[4].

The efficiency and accuracy of the proposed approach are verified through a representative example that employs the creep and relaxation models provided in the fib Model Code 2020 (MC2020) [5].

2 Integral-type creep formulation for concrete

The following integral-type creep law is obtained by applying Boltzmann's principle of superposition and considering concrete with linear behavior over time:

$$\varepsilon_c(t) = \int_{t'=0}^{t} J_c(t,t')\, d\sigma_c(t') \tag{1}$$

$\varepsilon_c(t)$ is the concrete strain at time t, $d\sigma_c(t')$ is the concrete stress increment at time t', and $J_c(t,t')$ is the creep compliance function from time t' to t.

For numerical implementation, the integral-type creep law can be reformulated into an incremental equation. To obtain the incremental form, both shrinkage and creep effects are incorporated, and the linearized stiffness of concrete $\bar{D}_c^i$ is introduced. This leads to

$$\Delta\varepsilon_c^i = \bar{D}_c^i \left\{ \Delta\sigma_c^i + \sum_{k=0}^{i-1} \Delta\sigma_c^k J_c\left(t^{i+1}, \Delta t^k\right) + \varepsilon_{c,sh}\left(t^{i+1}\right) - \varepsilon_{sh}^c\left(t^i\right) \right\} \tag{2}$$

where σ_c represents the concrete stress and ε_c^i is the concrete strain at time t_i. The stress and strain increments $\Delta\sigma_c^i = \sigma_c^{i+1} - \sigma_c^i$ and $\Delta\varepsilon_c^i = \varepsilon_c^{i+1} - \varepsilon_c^i$ are respectively defined in the time interval $\Delta t^i = t^{i+1} - t^i$. $J_c(t,\Delta t')$ is the creep compliance function at time t due to a load increment linearized at time interval $\Delta t'$. The shrinkage strain at time t is denoted as $\varepsilon_{c,sh}(t)$ and $\bar{D}_c^i$ is the linearized stiffness of concrete over the interval $\left[t^i, t^{i+1}\right]$, obtained as the inverse of $J_c\left[t^{i+1}, \Delta t^i\right]$.

Equation (2) expresses the incremental strain in concrete based on the discrete stress history and shrinkage evolution. The integral-type formulation requires storing and processing the full stress history at each iteration, making it computationally demanding.

3 Rate-type approximation for concrete creep

The rate-type creep law represents the creep compliance function of concrete through a discrete Kelvin chain. An elastic spring with modulus E_0 is connected in series with n_μ aging Kelvin units, each defined by an elastic modulus $E_\mu(t)$ and a viscosity $\eta_\mu(t)$ (Figure 1).

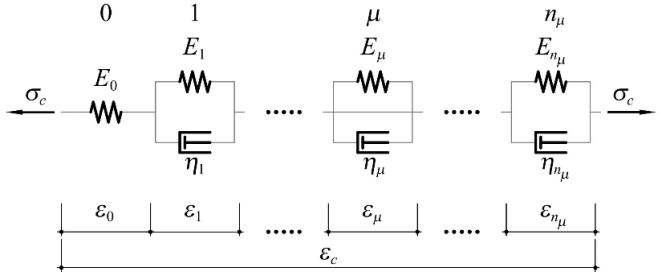


Figure 1. Kelvin chain model

The retardation time of each unit is given by $\tau_\mu = \eta_\mu(t) / E_\mu(t)$, where τ_μ and $\eta_\mu(t)$ are the retardation time and the viscosity of Kelvin unit μ, respectively. The total strain in concrete is given by the sum

$$\varepsilon_c(t) = \varepsilon_0(t) + \sum_{\mu=1}^{n_\mu} \varepsilon_\mu(t) \tag{3}$$

where $\varepsilon_0(t)$ is the strain in the isolated spring, $\varepsilon_\mu(t)$ is the strain of Kelvin unit μ, and n_μ is number of Kelvin units.

Based on the formulation by Bažant and Jirásek [2, 3], the compatibility and equilibrium in each Kelvin unit yields the following relation:

$$\dot{\sigma}_\mu(t) = E_\mu(t)\ \dot{\varepsilon}_\mu(t) + \eta_\mu(t)\ \ddot{\varepsilon}_\mu(t) \tag{4}$$

where $\dot{f}(t)$ is the time derivative of $f(t)$.

Assuming constant retardation time τ_μ, Equation (4) leads to the following second-order evolution equation:

$$\dot{\varepsilon}_\mu(t) + \tau_\mu\, \ddot{\varepsilon}_\mu(t) = \dot{\sigma}_c(t) / E_\mu(t) \tag{5}$$

By solving Equation (5) for pure creep under appropriate boundary conditions, the response to a constant stress $\bar{\sigma}_c(t')$ applied at time t' is given by

$$\varepsilon_\mu(t,t') = J_\mu(t,t')\ \bar{\sigma}_c(t') \tag{6}$$

where t and t' are the current and loading ages, respectively. All times are here expressed in days. The creep compliance function of unit μ is given by

$$J_\mu(t,t') = C_\mu(t')\left(1 - e^{-(t-t')/\tau_\mu}\right) \tag{7}$$

where $C_\mu(t') = 1 / E_\mu(t')$. An approximated creep function is then expressed as

$$\tilde{J}_c(t,t') = C_0(t') + \sum_{\mu=1}^{n_\mu} C_\mu(t')\left(1 - e^{-(t-t')/\tau_\mu}\right) \tag{8}$$

where $C_0(t') = 1 / E_0(t')$. $E_0(t')$ and $E_\mu(t')$ are here determined by fitting Equation (3) to a target creep compliance function, by using a least-squares method at selected points. This paper adopts the creep model of the fib Model Code 2020 [5].

The rate-type representation offers a numerically efficient approximation of creep behavior, making it well suited for large-scale structural analyses. An incremental formulation for the rate-type creep law can be derived using the superposition principle. The second-order differential equation (5), when integrated over the time interval Δt_i, leads to an expression for the strain increment in each Kelvin unit, without requiring accumulation of previous stress increments.

The resulting strain increment $\Delta\varepsilon_\mu^i$ in Kelvin unit μ is given by

$$\Delta\varepsilon_\mu^i = \bar{C}_\mu^i\, \Delta\sigma_c^i + \left(1 - e^{\xi_\mu^i}\right)\left(\varepsilon_\mu^\infty\left(t^i\right) - \varepsilon_\mu\left(t^i\right)\right) \tag{9}$$

where $\bar{C}_\mu^i$ is the effective compliance and $\varepsilon_\mu\left(t^i\right)$ is the internal strain at time t_i. $\xi_\mu^i = -\Delta t_i / \tau_\mu$ is the normalized time interval, with τ_μ being the retardation time of unit μ. ε_μ^∞ is the asymptotic strain defined as

$$\varepsilon_\mu^\infty\left(t^i\right) = \int_{t'=0}^{t^i} C_\mu(t')\, d\sigma_c(t') \tag{10}$$

Expressions for $\bar{C}_\mu^i$ and ε_μ^∞ are detailed in Reference [1].

The contribution of the isolated spring is given by

$$\Delta\varepsilon_0^i = \bar{C}_0^i \Delta\sigma_c^i \tag{11}$$

with $\bar{C}_0^i = \left(C_0\left(t^i\right) + C_0\left(t^{i+1}\right)\right)/2$.

The total strain increment in concrete, including shrinkage, results from summing all elements and applying the stiffness $\bar{D}_c^i$:

$$\bar{D}_c^i \, \Delta\varepsilon_c^i = \Delta\sigma_c^i + \bar{D}_c^i \left\{ \varepsilon_0\left(t^i\right) + \sum_{\mu=1}^{n_\mu} \left(e^{\xi_\mu^i} \, \varepsilon_\mu\left(t^i\right) + \left(1 - e^{\xi_\mu^i}\right) \varepsilon_\mu^\infty\left(t^i\right)\right) + \varepsilon_{c,sh}\left(t^{i+1}\right) - \varepsilon_c\left(t^i\right) \right\} \tag{12}$$

The linearized stiffness of concrete $\bar{D}_c^i$ is given by

$$\bar{D}_c^i = \left\{ \bar{C}_0^i + \sum_{\mu=1}^{n_\mu} \bar{C}_\mu^i \right\}^{-1} \tag{13}$$

The rate-type formulation avoids the explicit summation of past increments and improves computational efficiency.

4 Stress relaxation in prestressing steel under varying strain

The influence of varying strain histories on stress relaxation in prestressing steel is modeled using the approach proposed by Glodowski et al. [4], which assumes that strain remains nearly constant within each time interval. The stress at time t is given by

$$\sigma_p(t) = \left(1 - \rho(\bar{t}, \sigma_{p0})\right) \sigma_{p0} \tag{14}$$

where $\sigma_p(t)$ is the stress at time t, σ_{p0} is the maximum stress applied at time t_{p0}, and ρ is the relaxation function. Tensioning time $\bar{t} = t - t_{p0}$ is expressed in days.

Effective values are introduced to account for prior strain variations. The effective initial stress $\sigma_{p0,\mathrm{ef}}^i$ originates from σ_{p0}, but is adjusted to reflect the stress changes caused by strain variations up to the beginning of step i.

The corresponding effective time $\bar{t}_{\mathrm{ef}}^i$ is defined as the hypothetical tensioning time at which the stress would decrease from $\sigma_{p0,ef}^i$ to $\sigma_p\left(t^i\right)$ under pure relaxation. Since both $\sigma_{p0,ef}^i$ and $\sigma_p\left(t^i\right)$ are known at the start of step i, the following condition must be satisfied:

$$\rho\left(\bar{t}_{ef}^i\right) = \frac{\sigma_{p0,ef}^i - \sigma_p\left(t^i\right)}{\sigma_{p0,ef}^i} \tag{15}$$

From this condition, the effective time $\bar{t}_{\mathrm{ef}}^i$ is determined by solving:

$$\bar{t}_{ef}^i = \rho^{-1}\left(\rho\left(\bar{t}_{ef}^i\right)\right) \tag{16}$$

The incremental equation is expressed by

$$E_p \, \Delta\varepsilon_p^i = \Delta\sigma_p^i + r_p^i \tag{17}$$

where residual r_p^i is given by

$$r_p^i \simeq \left(\rho\left(\bar{t}_{ef}^i + \Delta t^i, \sigma_{p0,ef}^i\right) - \rho\left(\bar{t}_{ef}^i, \sigma_{p0,ef}^i\right)\right) \sigma_{p0,ef}^i \tag{18}$$

A detailed implementation of the relaxation function under varying strain conditions, following the MC2020 model [5], is provided in [1].

5 Cross-section analysis

The incremental formulation of the cross-section is derived assuming planar sections remain plane and that there is no slip between concrete and steel. The strain increment at each point is related to the generalized strain vector $\Delta\boldsymbol{\varepsilon}$ and the local coordinates x_1, x_2.

The internal force equilibrium for the concrete region is written as

$$\mathbf{K}_c^i \, \Delta\boldsymbol{\varepsilon}^i = \Delta\mathbf{f}_c^i + \mathbf{r}_c^i \tag{19}$$

where $\Delta\mathbf{f}_c^i$ and $\mathbf{r}_c^i$ are, respectively, the stress resultant increment and the residual vector for the concrete section. The corresponding stiffness matrix is

$$\mathbf{K}_c^i = \bar{D}_c^i \, \mathbf{G}_c \tag{20}$$

where

$$\mathbf{G}_c = \int_A \mathbf{x}\,\mathbf{x}^T \, dA \tag{21}$$

Residual vectors for integral- and rate-type concrete models are defined respectively by

$$\mathbf{r}_c^{i,I} = \mathbf{K}_c^i \left\{ \sum_{k=0}^{i-1} \left(J_c\left(t^{i+1}, \Delta t^k\right) \Delta\mathbf{f}_c^k \right) + \varepsilon_{c,sh}\left(t^{i+1}\right) - \varepsilon_c\left(t^i\right) \right\} \tag{22}$$

$$\mathbf{r}_c^{i,R} = \mathbf{K}_c^i \left\{ \varepsilon_0\left(t^i\right) + \sum_{\mu=1}^{n_\mu} \left(\varepsilon_\mu^\infty\left(t^i\right) + e^{\varphi_\mu^i} \left(\varepsilon_\mu\left(t^i\right) - \varepsilon_\mu^\infty\left(t^i\right) \right) \right) + \varepsilon_{c,sh}\left(t^{i+1}\right) - \varepsilon_c\left(t^i\right) \right\} \tag{23}$$

The contribution from prestressing steel is

$$\mathbf{K}_p \, \Delta\boldsymbol{\varepsilon}^i = \Delta\mathbf{f}_p^i + \mathbf{r}_p^i \tag{24}$$

where $\mathbf{K}_p^i$, $\Delta\mathbf{f}_p^i$, and $\mathbf{r}_p^i$ are the stiffness matrix, the stress resultant increment, and the residual vector of a prestressing steel tendon, respectively.

The full equilibrium of the cross-section is then

$$\mathbf{K}^i \, \Delta\boldsymbol{\varepsilon}^i = \Delta\mathbf{f}^i + \mathbf{r}^i \tag{25}$$

with

$$\mathbf{K}^i = \mathbf{K}_c^i + \sum \mathbf{K}_p + \sum \mathbf{K}_s^i \tag{26}$$

$$\mathbf{r}^i = \mathbf{r}_c^i + \sum \mathbf{r}_p^i \tag{27}$$

$$\Delta\mathbf{f}^i = \Delta\mathbf{f}_c^i + \sum \Delta\mathbf{f}_p^i + \sum \Delta\mathbf{f}_s^i \tag{28}$$

6 Example

To illustrate the application of the proposed formulation, a group of pre-tensioned concrete beams analyzed by Reybrouck, Van Mullem, Taerwe, and Caspeele [6] is revisited. These authors evaluated long-term experimental data of prestressed members using a calculation framework originally proposed by Gilbert and Ranzi[7], which provided excellent agreement with experimental results when applied in conjunction with EN 1992-1-1:2004 [8] and the fib Model Code 2010 (MC2010) [9].

The midspan deflections of six pre-tensioned beams (type I-S) from that study are recalculated using the proposed rate-type incremental formulation. Sectional analysis results are integrated along the beam length using cubic interpolation over 500 mm segments. Gravel is reported as the aggregate type and is assumed to originate from limestone. Shrinkage is considered from the start of curing ($t_s = 0$), and the stress relaxation parameter is adjusted to match the relaxation losses observed at 5,000 hours. Geometrical and material properties are provided in Reference [1].

The six beams are grouped by prestress transfer times ($t_p = 7$ and 14 days). For each group, different load combinations ($Q_0 = 0$, $Q_1 = 27.45$ kN, and $Q_2 = 54.90$ kN) are applied at 1/4 and 3/4 span positions at $t_Q = 28$ days. Time intervals are defined on a logarithmic scale (1, 2, 4, … days) and restart at each new loading event. The small number of time steps—only 18 and 17 for the two groups, respectively—demonstrates the computational efficiency of the proposed approach, while still yielding results in close agreement with experimental data.

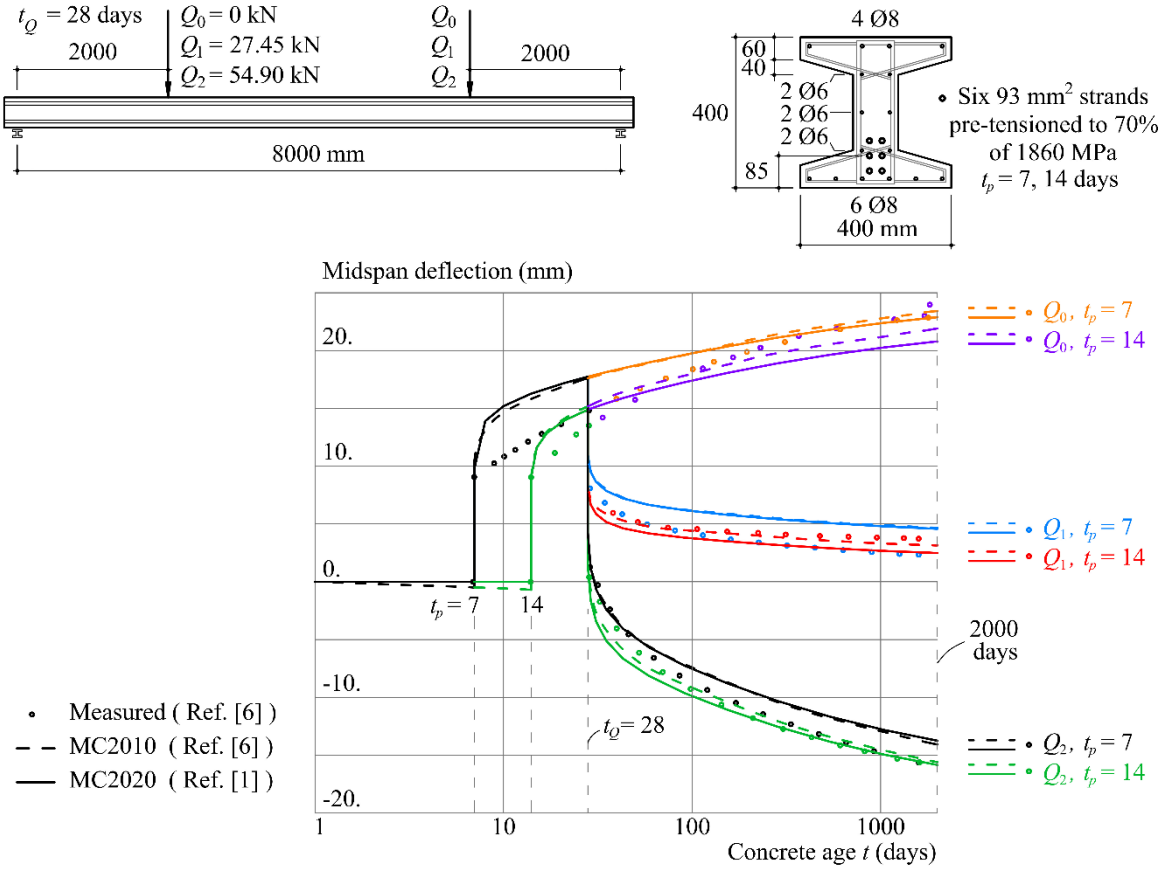


Figure 2. Midspan deflections of beams of type I-S (Reference [1])

The computed midspan deflections are shown in Figure 2. The results closely match those obtained by Reybrouck et al. [6] using the MC2010-based approach and are in good agreement with the experimental data from the Belgian testing program. While some discrepancies between the computed and experimental deflections are observed between t_p and t_Q , both formulations provide similar responses throughout this interval and capture the long-term behavior with comparable accuracy.

Additional examples of reinforced, pre-tensioned, and post-tensioned concrete beams analyzed using both the integral- and rate-type approaches are provided in Reference [1], where relative deviations remain below 0.1% throughout the time history.

7 Conclusions

This study presents a general rate-type formulation for the time-dependent analysis of reinforced and prestressed concrete structures. The incremental equation is derived directly from the rate-type creep law, without introducing higher-order differential terms. Compliance values are linearly interpolated within each time interval to improve accuracy, while preserving consistency with the method originally proposed by Bažant and Jirásek [2, 3] for aging materials.

Numerical examples confirm the close agreement between the integral- and rate-type solutions, with relative differences below 0.1% over extended time spans. These results validate the proposed approach and demonstrate its suitability for practical applications.

Rate-type formulations offer clear advantages in terms of efficiency, especially in large-scale simulations, where storing and updating full stress and strain histories becomes computationally demanding.

For a complete derivation and detailed discussion, the reader is referred to [1].

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References

[1] M. Schulz, "Rate-type and integral-type sectional analysis for time-dependent response with the fib Model Code 2020," *Structural Concrete,* Early View, doi: https://doi.org/10.1002/suco.70035.
[2] Z. P. Bažant and M. Jirásek, *Creep and hygrothermal effects in concrete structures*, 1 ed. (Solid Mechanics and Its Applications, no. 225). Dordrecht, Netherlands: Springer, 2018.
[3] M. Jirásek and Z. P. Bažant, *Inelastic analysis of structures*, 1 ed. Chichester, West Sussex, England: John Wiley & Sons, 2002.
[4] R. Glodowski, G. Hoff, and A. Fox, "Stress relaxation of steel tendons used in prestressed concrete under conditions of changing applied stress," in *Stress Relaxation Testing*, vol. STP676-EB. Philadelphia, PA: ASTM, 1979, pp. 42–58.
[5] fib (International Federation for Structural Concrete), *fib Model Code for Concrete Structures (2020)*. Lausanne: fib, 2023.
[6] N. Reybrouck, T. Van Mullem, L. Taerwe, and R. Caspeele, "Influence of long-term creep on prestressed concrete beams in relation to deformations and structural resistance: Experiments and modeling," *Structural Concrete,* vol. 21, no. 4, pp. 1458-1474, 2020, doi: https://doi.org/10.1002/suco.201900418.
[7] R. I. Gilbert and G. Ranzi, *Time-dependent behaviour of concrete structures*, 1st ed. ed. London: CRC Press, 2011.
[8] CEN (Comité Européen de Normalisation), *Eurocode 2: Design of concrete structures - General rules and rules for buildings*. Brussels: CEN, 2004.
[9] fib (International Federation for Structural Concrete), *fib Model Code for Concrete Structures 2010*. Berlin, Germany: Ernst & Sohn, 2013.